

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Exam in: FYS3120 Classical Mechanics and Electrodynamics

Day of exam: 12 June 2024

Exam hours: 15.00 (4 hours)

This examination paper consists of 5 (five) pages.

Permitted materials: Approved calculator
Rottman: "Matematisk formelsamling"
Compendium with formulas for the course: "FYS3120
Classical Mechanics and Electrodynamics Collection of
formulae 2024"

*Make sure that your copy of this examination paper
is complete before answering.*

You may answer the questions either in English or Norwegian.

*The exam has a maximum possible score of 80 points. Up to four points will be
awarded for clear, concise and well argued answers.*

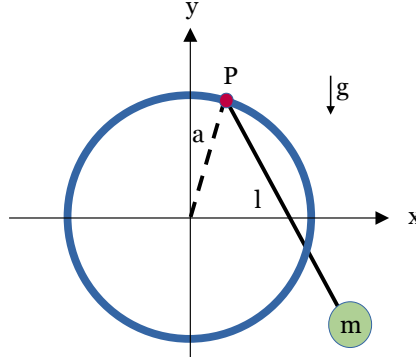


Figure 1: Pendulum mass attached to a rotating wheel.

Question 1 A very driven pendulum [25 points]

A stiff pendulum of length ℓ and mass m is attached to a pivot point P on a wheel with radius a . The wheel is driven by a motor to rotate with constant angular velocity ω in the clockwise direction. Gravity works in the negative y -direction. There is no friction in the system and all motion is assumed to be in the plane. The system is illustrated in Fig. [1](#).

The wheel is started so that at $t = 0$ the pivot point (x_P, y_P) is at the top of the wheel, $(x_P, y_P) = (0, a)$.

- a) Find the number of degrees of freedom and define generalised coordinate(s) for the system. [3 points]
- b) Show that the Lagrangian for the system can be written as

$$L = \frac{1}{2}m[\ell^2\dot{\theta}^2 + 2a\ell\omega\dot{\theta}\cos(\theta + \omega t) + a^2\omega^2] - mg(a\cos\omega t - \ell\cos\theta). \quad (1)$$

[6 points]

- c) Explain why the Lagrangian

$$L = \frac{1}{2}\dot{\theta}^2 + \frac{a\omega^2}{\ell}\cos(\theta - \omega t) + \frac{g}{\ell}\cos\theta, \quad (2)$$

is equivalent to the one found in **b)**, meaning it has the same equations of motion. Do this without finding the equations of motion. [3.5 points]

- d) Find the equations of motion. [3.5 points]
- e) Show that in the limit of fast rotation of the wheel, $\omega \gg \sqrt{\frac{g}{a}}$, we have the solution $\theta(t) = \omega t$. [2 points]

- f) If we look at small perturbations around this fast solution, defining $\phi(t) \equiv \theta(t) - \omega t$, where $\phi \ll 1$, show that the solution for ϕ is the harmonic oscillator like solution $\phi(t) = \phi_0(t) \sin \omega t$, and find $\phi_0(t)$. Comment on what happens when ℓ and a are similar in size. [4 points]
- g) Given the transformation of a Lagrangian $L(q, \dot{q}, t)$,

$$L \rightarrow L' = aL + b + \frac{d}{dt}f(q, t), \quad (3)$$

how does the Hamiltonian $H(q, p, t)$ transform? [3 points]

Question 2 Muon field trip I [28 points]

A muon is an unstable particle, which can be viewed as a more massive cousin of the electron. In this problem we study a muon which is created in the laboratory at $t = 0$ with $\mathbf{v} = 0$, and is then accelerated by a constant – in time and space – electric field $\mathbf{E}$, while there is no magnetic field (as measured in the laboratory reference frame). The mass of the muon is m and its electric charge e . The lifetime of the muon in its rest frame is τ_0 .

For now, we will disregard the effect of the radiation coming from the acceleration of the muon.

- a) Use the relativistic form of Newton's second law

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \quad (4)$$

to find the relativistic momentum of the muon $\mathbf{p}(t)$ in the laboratory reference frame. [2 points]

- b) Find the relativistic energy $\mathcal{E}(t)$ and show that the γ -factor for the muon in the laboratory frame is¹

$$\gamma(t) = \sqrt{1 + k^2 t^2}, \quad \text{where} \quad k = \frac{eE}{mc}.$$

[3 points]

- c) Show that the lifetime t_0 of the muon measured in the laboratory is

$$t_0 = \frac{1}{k} \sinh(k\tau_0). \quad (5)$$

[4 points]

¹We use $\mathcal{E}$ for the energy here so as not to confuse it with the electric field.

- d) Show that the relativistic energy in the laboratory frame, as a function of the proper time τ , is

$$\mathcal{E}(\tau) = mc^2 \cosh(k\tau). \quad (6)$$

[2 points]

- e) Show that the four-acceleration of the muon A^μ can be expressed in the laboratory frame as

$$A^\mu = c(k^2 t, \gamma k \hat{\mathbf{E}}). \quad (7)$$

[5 points]

- f) Explain what the instantaneous inertial rest frame of the muon is. Show that the proper acceleration of the muon $\mathbf{a}_0$ is a constant vector in this reference frame given by

$$\mathbf{a}_0 = kc\hat{\mathbf{E}}, \quad (8)$$

where $\hat{\mathbf{E}}$ is the unit vector in the direction of the electric field, by using the four-acceleration² and explain why the motion is linear. [5 points]

- g) What are the electric and magnetic fields observed by the muon in its rest frame? [3.5 points]

- h) Show that the acceleration in the laboratory frame is

$$\mathbf{a} = \frac{\mathbf{a}_0}{\gamma^3}. \quad (9)$$

[3.5 points]

Question 3 Muon field trip II [23 points]

We now look at the radiation from the muon in the previous question.

- a) Find the electric dipole moment of the muon as observed in the laboratory in terms of its path $\mathbf{r}(t)$ measured from the origin. [3 points]
- b) Find the magnetic dipole moment of the muon as observed in the laboratory. [4 points]
- c) Using the Liénard-Wiechert potentials one can show³ that the radiation part of the fields for a relativistic particle far from the observer can be written in terms of the electric dipole moment $\mathbf{p}$ as

$$\mathbf{E}_{\text{rad}}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r} \left[\frac{\ddot{\mathbf{p}} \times (\hat{\mathbf{n}} - \frac{\mathbf{v}}{c})}{(1 - \hat{\mathbf{n}} \cdot \frac{\mathbf{v}}{c})^3} \right]_{\text{ret}} \times \hat{\mathbf{n}}, \quad (10)$$

$$\mathbf{B}_{\text{rad}}(\mathbf{r}, t) = \frac{1}{c} \hat{\mathbf{n}} \times \mathbf{E}_{\text{rad}}(\mathbf{r}, t), \quad (11)$$

²There is another way of doing this, but that will not score any points. Sorry.

³Since we are decent people we won't torture you with the details on this exam.

where $\hat{\mathbf{n}}$ is the unit vector in the direction of the observer and “ret” signifies evaluation at the retarded time $t_r = t - r/c$.

Use this to find the radiation fields $\mathbf{E}_{\text{rad}}(\mathbf{r}, t)$ and $\mathbf{B}_{\text{rad}}(\mathbf{r}, t)$ for the muon in the laboratory frame. Comment on the direction of the fields with respect to the observer. *Hint:* Try to remove the velocity to leave only k and $\hat{\mathbf{E}}$ from Question 2 as free parameters apart from the standard constants. We will give points for partial answers. [6 points]

- d) Find Poynting’s vector for the radiation fields and comment on its angular dependence. [6 points]
- e) Power P can be shown to be a Lorentz invariant quantity. Starting from the non-relativistic form found in the formulae collection, use this to argue why the relativistic form of Larmor’s formula giving the radiated power from single charged particle of charge q is

$$P = \frac{\mu_0 q^2}{6\pi c} \mathbf{a}_0^2, \quad (12)$$

where $\mathbf{a}_0$ is the proper acceleration. Use this formula to show that the total energy the muon radiates in its lifetime t_0 , as measured in the laboratory, is

$$\mathcal{E}_{\text{rad}} = \frac{\mu_0 e^2}{6\pi} k c \sinh(k\tau_0). \quad (13)$$

[4 points]