

Question 1 Mexican hat [21 points]

We begin by considering a particle moving in the one-dimensional Mexican hat potential¹

$$V(x) = -\frac{1}{2}kx^2 + \frac{1}{4}\lambda x^4,$$

shown in Fig. 1, where k and λ are positive parameters. The corresponding Lagrangian is²

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 - \frac{1}{4}\lambda x^4, \quad (1)$$

where x is the generalised coordinate of the system and $\dot{x} = \frac{dx}{dt}$.

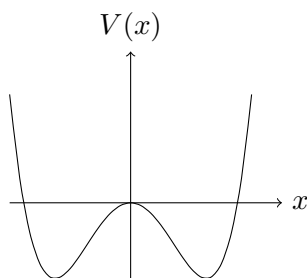


Figure 1: The Mexican hat potential. The name is given from the not so obvious similarity to the famous Mexican sombrero.

- a) Find and classify the equilibrium points of the system. [4 points]
- b) Find the equation of motion for the system. [4 points]
- c) Find the Hamiltonian. [4 points]
- d) Find one continuous symmetry of this Lagrangian and what is conserved under that symmetry. [3 points]

Suppose now that the particle is moving in the two-dimensional Mexican hat potential shown in Fig. 2, where the Lagrangian can be written

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}k(x^2 + y^2) - \frac{1}{4}\lambda(x^2 + y^2)^2. \quad (2)$$

- e) Show that this Lagrangian is symmetric under a rotation in the xy -plane. [3 points]

¹Since it is one dimensional it is a bit hard to wear, but it works fine mathematically.

²This is a mechanical hat, not a field theory hat, so it is not the problem you studied in Problem Set 12, but it has some distinct similarities.

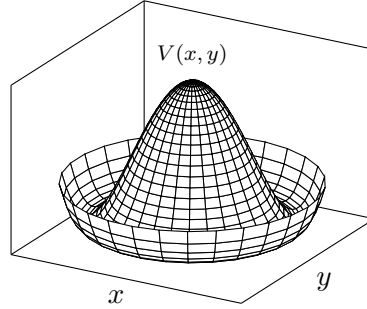


Figure 2: The two-dimensional Mexican hat potential.

- f) Find and identify the conserved Noether charge associated with the rotation. *Hint:* Write the rotation in terms of an infinitesimal angle $\delta\theta$. [3 points]

Question 2 Some action [19 points]

Consider the action

$$S[x(\tau)] = -mc^2 \int_{\tau_0}^{\tau_1} d\tau, \quad (3)$$

where τ is the proper time for a particle moving in one space dimension x .

- a) Explain the meaning of proper time. [2 points]
b) Show that the Lagrangian for the particle in some reference frame with coordinates x and time t is

$$L(x, \dot{x}) = -mc^2 \left(1 - \frac{\dot{x}^2}{c^2} \right)^{1/2}.$$

[3 points]

- c) Find the equation of motion for the particle and show that it reduces to the equation of motion of a free non-relativistic particle. [5 points]
d) Show directly from the Lorentz transformation

$$t' = \gamma \left(t - \frac{vx}{c^2} \right), \quad x' = \gamma(x - vt), \quad (4)$$

that the action in (3) is invariant under Lorentz transformations. Are the equations of motion also invariant? *Hint:* The invariant line element is $ds = \sqrt{c^2 dt^2 - dx^2}$ in one space dimension. [5 points]

- e) Now, consider a more general expression for an action

$$S[x(\lambda)] = -M \int_{\lambda_0}^{\lambda_1} \frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda} d\lambda, \quad (5)$$

where the motion described in terms of the position four-vector $x^\mu(\lambda)$ is parameterised by some parameter λ , and where M is some constant. Under what conditions will this be equivalent to the action in Eq. (3)? [4 points]

Question 3 How Lorentz found the transformations [19 points]

We will look at a particle with charge q moving at constant velocity v in the z -direction through the origin. We want to find the potentials which are observed in the xy -plane as a result of the charge.

We remind you that the **Liénard-Wiechert potentials** for a point charge with path $\mathbf{r}(t)$ and velocity $\mathbf{v}(t)$ can be written

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{(R - \boldsymbol{\beta} \cdot \mathbf{R})_{\text{ret}}}, \quad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 q c}{4\pi} \left(\frac{\boldsymbol{\beta}}{R - \boldsymbol{\beta} \cdot \mathbf{R}} \right)_{\text{ret}},$$

where $\mathbf{R}(t) = \mathbf{r} - \mathbf{r}(t)$ is the vector between the observer and the particle, $\boldsymbol{\beta} = \frac{\mathbf{v}(t)}{c}$, and “ret” indicates that the time of evaluation is the retarded time

$$t_- = t - \frac{1}{c} R(t_-) = t - \frac{1}{c} |\mathbf{r} - \mathbf{r}(t_-)|.$$

- a) Explain what the retarded time is, and show that for an observer in the xy -plane it can be expressed as

$$t_- = t - \frac{1}{c} \sqrt{x^2 + y^2 + v^2 t_-^2}. \quad (6)$$

[5 points]

- b) Solve for the retarded time in Eq. (6) to show that

$$t_- = \gamma^2 t - \frac{\gamma^2}{c} \sqrt{v^2 t^2 + (x^2 + y^2)/\gamma^2}. \quad (7)$$

[4 points]

- c) Show that the scalar and vector potentials in the xy -plane can be written as

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{(x^2 + y^2)/\gamma^2 + v^2 t^2}}, \quad (8)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{q\mu_0}{4\pi} \frac{v\mathbf{e}_z}{\sqrt{(x^2 + y^2)/\gamma^2 + v^2 t^2}}. \quad (9)$$

[6 points]

- d) Compare the result for the scalar potential in Eq. (8) to the scalar potential and electric field for a static charge at the origin

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{\sqrt{x^2 + y^2 + z^2}}, \quad \mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \mathbf{r}. \quad (10)$$

Why are these expressions different, and can you explain the differences? [4 points]

Question 4 Sine-Gordon equation [17 points]

The **sine-Gordon Lagrangian** is given by³

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - m^2(1 - \cos \phi). \quad (11)$$

- a) Show that the resulting equation of motion, the **sine-Gordon equation**, is

$$\square \phi + m^2 \sin \phi = 0. \quad (12)$$

[4 points]

- b) Find the generalised momentum density and the Hamiltonian density. [4 points]

- c) When is energy conserved for a field theory? Discuss whether energy is conserved for this model. [5 points]

- d) Show that in one space dimension, where $\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$,

$$\phi_s(x) = 4 \arctan \xi, \quad (13)$$

is a solution to the sine-Gordon equation with $\xi = e^{m\gamma(x-vt)+\delta}$, where v is the (constant) velocity of the wave in the x -direction, γ is the standard relativistic factor that depends on v , and δ a constant.

Hint:

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}, \quad \sin(4 \arctan x) = \frac{4x(1-x^2)}{(1+x^2)^2}. \quad (14)$$

Comment: ϕ_s is called a **soliton** solution. [4 points]

³It's a joke. The name, not the exam question. It's almost the Klein-Gordon equation, except with a sine.