

FYS3120 Analytical Mechanics and Electrodynamics

Collection of formulae*

1 Analytical Mechanics

Standard expressions for the Lagrangian and Hamiltonian

$$L = K - V, \quad (1)$$

$$H = K + V, \quad (2)$$

with kinetic energy K (denoted as T in the book) and potential energy V . Note that cases exist where H is *not* the total energy.

The Lagrangian

$$L = L(q, \dot{q}, t), \quad (3)$$

is a function of the *generalised coordinates* $q = \{q_i; i = 1, 2, \dots, d\}$ of the system, and their time derivatives $\dot{q} = \{\dot{q}_i; i = 1, 2, \dots, d\}$. The Lagrangian may also have an *explicit* dependence on time t .

Lagrange's equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \dots, d, \quad (4)$$

are the resulting *equations of motion* (e.o.m.). There is one equation for each generalised coordinate.

Generalised momentum

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \quad i = 1, 2, \dots, d. \quad (5)$$

is also referred to as *canonical* or *conjugate* momentum. There is one generalised momentum p_i conjugate to each generalised coordinate q_i .

The Hamiltonian

$$H(p, q) \equiv \sum_{i=1}^d \dot{q}_i p_i - L, \quad (6)$$

is considered as a function of the generalised coordinates q_i and momenta p_i .

*Version 13/01/2026. Very closely following earlier version by Are Raklev.

Hamilton's equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \dots, d, \quad (7)$$

are the equations of motion from the Hamiltonian $H(p, q)$.

Poisson bracket

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right), \quad (8)$$

for functions $f(q, p)$ and $g(q, p)$ of the generalised coordinates and momentum.

Lagrangian and Hamiltonian for a charged particle in an electromagnetic field

$$L = L(\mathbf{r}, \mathbf{v}) = \frac{1}{2}mv^2 - q\phi + q\mathbf{v} \cdot \mathbf{A}, \quad (9)$$

$$H = H(\mathbf{r}, \mathbf{p}) = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi. \quad (10)$$

Here $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ are the scalar and vector potentials. q is the charge of the particle. These equations are non-relativistic.

Noether charge

$$\delta Q \equiv \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i - \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right) \delta t - \delta G, \quad (11)$$

is the quantity that is conserved if the e.o.m. are form-invariant under the infinitesimal transformation of the generalised coordinates δq_i , and time δt . Here δG denotes a possible term from a total time-derivative in the transformed Lagrangian (also expanded in δq_i and δt).

2 Relativity

Space-time coordinates as a *contravariant* four-vector

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z) = (ct, \mathbf{r}). \quad (12)$$

The general Lorentz transformation

$$x^\mu \rightarrow x'^\mu = L^\mu_\nu x^\nu, \quad (13)$$

where L^μ_ν is the Lorentz transformation tensor.

Special case of Lorentz transformation with velocity v in the x -direction

$$\begin{aligned} x'^0 &= \gamma(x^0 - \beta x^1), \\ x'^1 &= \gamma(x^1 - \beta x^0), \\ x'^2 &= x^2, \\ x'^3 &= x^3, \end{aligned}$$

with $\beta = v/c$ and $\gamma = 1/\sqrt{1 - \beta^2}$.

Condition satisfied by Lorentz transformation matrices

$$g_{\mu\nu} L^\mu_\rho L^\nu_\sigma = g_{\rho\sigma}. \quad (14)$$

Invariant line element

$$\Delta s^2 = c^2 \Delta t^2 - \Delta \mathbf{r}^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu = \Delta x_\mu \Delta x^\mu. \quad (15)$$

Metric tensor

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad (16)$$

is the metric tensor for flat Minkowski space.

Raising and lowering of indices

The index of a four-vector can be lowered and raised using the metric tensor. The *covariant* space-time four-vector is

$$x_\mu = g_{\mu\nu} x^\nu, \quad x_\mu = (ct, -\mathbf{r}). \quad (17)$$

We also have

$$x^\mu = g^{\mu\nu} x_\nu, \quad g_{\mu\rho} g^{\rho\nu} = \delta_\mu^\nu, \quad (18)$$

where δ_μ^ν is the *Kronecker-delta*. ($\delta_\mu^\nu = 0$ if $\mu \neq \nu$, and $\delta_\mu^\nu = 1$ if $\mu = \nu$.)

Proper time – time dilatation

$$d\tau = \frac{1}{c} \sqrt{ds^2} = \frac{1}{\gamma} dt, \quad (19)$$

Here $d\tau$ is the proper time interval, meaning the time measured in the (instantaneous) rest frame of a moving body by a co-moving clock, ds^2 is the invariant line element of an infinitesimal section of the object's world line, and dt is the coordinate time interval, meaning the time interval measured in arbitrarily chosen inertial reference frame of velocity v with respect to the body.

Length contraction

$$L = \frac{1}{\gamma} L_0 \quad (20)$$

Lengths of a body measured in the direction of its motion. L_0 is the length measured in the rest frame of the body, L is the length measured (at simultaneity) in an arbitrarily chosen inertial frame.

Four-velocity

$$U^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma (c, \mathbf{v}). \quad (21)$$

This has the property that the contraction $U^\mu U_\mu = c^2$.

Four-acceleration

$$A^\mu \equiv \frac{dU^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2} = \gamma^2 \left(\gamma^2 \frac{\mathbf{v} \cdot \mathbf{a}}{c}, \mathbf{a} + \gamma^2 \frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \mathbf{v} \right). \quad (22)$$

This has the property that the contraction $A^\mu U_\mu = 0$.

Proper acceleration

The proper acceleration $\mathbf{a}_0$ is the acceleration measured in the instantaneous rest frame. We then have

$$A^\mu A_\mu = -\mathbf{a}_0^2. \quad (23)$$

Relativistic energy and momentum

$$E \equiv \gamma m c^2, \quad (24)$$

$$\mathbf{p} \equiv \gamma m \mathbf{v}. \quad (25)$$

Four-momentum

$$p^\mu = m U^\mu = m \gamma (c, \mathbf{v}) = \left(\frac{E}{c}, \mathbf{p} \right), \quad (26)$$

where m is the mass of the moving body.

Four-derivative

$$\partial_\nu \equiv \frac{d}{dx^\nu} = \left(\frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right), \quad (27)$$

and $\partial^\mu = g^{\mu\nu} \partial_\nu$.

3 Electrodynamics

Maxwell's equations in differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (28)$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial}{\partial t} \mathbf{E} = \mu_0 \mathbf{j}, \quad (29)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (30)$$

$$\nabla \times \mathbf{E} + \frac{\partial}{\partial t} \mathbf{B} = 0. \quad (31)$$

Electromagnetic potentials

$$\mathbf{E} = -\nabla \phi - \frac{\partial}{\partial t} \mathbf{A}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (32)$$

Lorentz force

Force from the electromagnetic field on a point particle with charge q

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (33)$$

Four-potential

$$A^\mu \equiv \left(\frac{\phi}{c}, \mathbf{A} \right). \quad (34)$$

Four-current density

$$j^\mu \equiv (c\rho, \mathbf{j}). \quad (35)$$

Electromagnetic field strength tensor

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu, \quad (36)$$

where

$$F^{k0} = \frac{1}{c} E_k, \quad F^{ij} = -\epsilon_{ijk} B_k. \quad (37)$$

The *dual tensor* is

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad (38)$$

where

$$\tilde{F}^{k0} = B_k, \quad \tilde{F}^{ij} = \frac{1}{c} \epsilon_{ijk} E_k. \quad (39)$$

Maxwell's equations in covariant form

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0, \quad (40)$$

Charge conservation

$$\partial_\mu j^\mu = 0, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0. \quad (41)$$

Potentials from charge and current distributions

in Lorenz gauge, $\partial_\mu A^\mu = 0$:

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} dV', \quad (42)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} dV'. \quad (43)$$

Retarded time

$$t' \equiv t - \frac{1}{c} |\mathbf{r} - \mathbf{r}'|. \quad (44)$$

Electric dipole moment

$$\mathbf{p} \equiv \int \mathbf{r} \rho(\mathbf{r}) dV. \quad (45)$$

Electric dipole potential (dipole at the origin)

$$\phi = \frac{\mathbf{n} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2}, \quad \mathbf{n} = \frac{\mathbf{r}}{r}. \quad (46)$$

Force and torque (about the origin)

$$\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E}, \quad \mathbf{M} = \mathbf{p} \times \mathbf{E}. \quad (47)$$

Magnetic dipole moment

$$\mathbf{m} \equiv \frac{1}{2} \int \mathbf{r} \times \mathbf{j}(\mathbf{r}) dV. \quad (48)$$

Magnetic dipole potential (dipole at the origin)

$$\mathbf{A} = \frac{\mu_0}{4\pi r^2} \mathbf{m} \times \mathbf{n}, \quad \mathbf{n} = \frac{\mathbf{r}}{r}. \quad (49)$$

Force and torque (about the origin)

$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B}) \text{ (current loop)}, \quad \mathbf{M} = \mathbf{m} \times \mathbf{B}. \quad (50)$$

Lorentz transformation of the electromagnetic field

$$F'^{\mu\nu} = L^\mu_\rho L^\nu_\sigma F^{\rho\sigma}. \quad (51)$$

Lorentz transformations for electric and magnetic fields

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel}, \quad \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}), \quad (52)$$

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel}, \quad \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \mathbf{v} \times \mathbf{E}/c^2). \quad (53)$$

The fields are decomposed in a parallel component ($\parallel$), in the direction of transformation velocity $\mathbf{v}$, and a perpendicular component ($\perp$), orthogonal to $\mathbf{v}$.

Lorentz invariants for electromagnetic fields

$$\mathbf{E}^2 - c^2 \mathbf{B}^2 = -\frac{c^2}{2} F_{\mu\nu} F^{\mu\nu}, \quad \mathbf{E} \cdot \mathbf{B} = \frac{c}{4} \tilde{F}_{\mu\nu} F^{\mu\nu}. \quad (54)$$

Electromagnetic field energy density

$$u = \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2) = \frac{\epsilon_0}{2}(E^2 + c^2 B^2). \quad (55)$$

Electromagnetic energy current density (Poynting's vector)

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}. \quad (56)$$

Monochromatic plane waves, plane polarised

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t), \quad \mathbf{E}_0 = E_0 \mathbf{e}_1, \quad (57)$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t), \quad \mathbf{B}_0 = B_0 \mathbf{e}_2, \quad (58)$$

$$\mathbf{E}_0 \cdot \mathbf{k} = \mathbf{B}_0 \cdot \mathbf{k} = 0, \quad \mathbf{B}_0 = \frac{1}{c} \mathbf{n} \times \mathbf{E}_0, \quad \mathbf{n} = \frac{\mathbf{k}}{k}. \quad (59)$$

Monochromatic plane waves, circular polarised

$$\mathbf{E}(\mathbf{r}, t) = \text{Re}(\mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]) , \quad \mathbf{E}_0 = E_0 \frac{1}{\sqrt{2}}(\mathbf{e}_1 \pm i\mathbf{e}_2), \quad (60)$$

$$\mathbf{B}(\mathbf{r}, t) = \text{Re}(\mathbf{B}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]) , \quad \mathbf{B}_0 = B_0 \frac{1}{\sqrt{2}}(\mathbf{e}_2 \mp i\mathbf{e}_1). \quad (61)$$

Polarisation vectors

$$\mathbf{e}_1 \cdot \mathbf{k} = \mathbf{e}_2 \cdot \mathbf{k} = 0, \quad \mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \quad \mathbf{e}_1^2 = \mathbf{e}_2^2 = 1. \quad (62)$$

Four-wave vector

$$k^\mu \equiv \left(\frac{\omega}{c}, \mathbf{k} \right), \quad \omega = ck. \quad (63)$$

Radiation fields, in the wave zone ($r \gg r', \lambda$)

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi c} \frac{\mathbf{n}}{r} \times \frac{d}{dt} \int \mathbf{j}(\mathbf{r}', t') dV' \quad \mathbf{n} = \frac{\mathbf{r}}{r}, \quad (64)$$

$$(65)$$

$$\mathbf{E}(\mathbf{r}, t) = c\mathbf{B}(\mathbf{r}, t) \times \mathbf{n}. \quad (66)$$

Electric dipole radiation

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi c} \frac{\mathbf{n}}{r} \times \ddot{\mathbf{p}}(t - r/c), \quad \mathbf{E}(\mathbf{r}, t) = c\mathbf{B}(\mathbf{r}, t) \times \mathbf{n}. \quad (67)$$

Radiation from accelerated, charged particle

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi c r} [\mathbf{a} \times \mathbf{n}]_{ret}, \quad \mathbf{E}(\mathbf{r}, t) = c\mathbf{B}(\mathbf{r}, t) \times \mathbf{n}_{ret}, \quad (68)$$

$$\mathbf{n} = \mathbf{R}/R, \quad \mathbf{R}(t) = \mathbf{r} - \mathbf{r}(t), \quad (69)$$

where $\mathbf{r}(t)$ is the vector showing the particle's path.

Radiated power, Larmor's formula

$$P = \frac{\mu_0 q^2}{6\pi c} \mathbf{a}^2, \quad (70)$$

where $\mathbf{a}$ is the acceleration of the charge q .

4 Relativistic field theory

The Lagrangian density of a field

$$\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi, x), \quad \text{with} \quad L \equiv \int \mathcal{L} d^3r, \quad (71)$$

is a function of the field ϕ (the generalised coordinate) and the covariant derivative $\partial_\mu \phi$. The Lagrangian density may also have an explicit dependence on the coordinates x^μ .

Lagrange's equations

$$\frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \quad (72)$$

are the e.o.m. for the field ϕ .

The generalised momentum density for ϕ

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}. \quad (73)$$

We also define

$$\pi^\mu(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}. \quad (74)$$

Hamiltonian density

$$\mathcal{H}(\phi, \nabla \phi, \pi, x) \equiv \pi(x) \partial_0 \phi(x) - \mathcal{L}(\phi, \partial_\mu \phi, x), \quad \text{with} \quad H \equiv \int \mathcal{H} d^3r, \quad (75)$$

The Noether four-current

$$\delta N^\mu \equiv \pi^\mu (\delta \phi - \partial_\nu \phi \delta x^\nu) + \mathcal{L} \delta x^\mu - \delta G^\mu, \quad (76)$$

is the quantity that is conserved if the e.o.m. are form invariant under the infinitesimal transformation of the field $\delta \phi$ and coordinates δx^μ . Here δG is a possible term from a total derivative term in the transformed Lagrangian. This should be understood in the sense that the Noether current fulfils a continuity equation

$$\partial_\mu \delta N^\mu = 0. \quad (77)$$

The canonical energy-momentum tensor for ϕ

$$T^{\mu\nu} = \pi^\mu \partial^\nu \phi - \mathcal{L} g^{\mu\nu}, \quad (78)$$

is conserved if the e.o.m. are invariant under translation of four-vector coordinates,

$$x^\mu \rightarrow x'^\mu = x^\mu + a^\mu, \quad (79)$$

in the sense $\partial_\mu T^{\mu\nu} = 0$.

The (symmetric) energy-momentum tensor for an electromagnetic field

$$T^{\mu\nu} = \frac{1}{\mu_0} \left(-F^{\mu\rho} F^\nu{}_\rho + \frac{1}{4} F^{\rho\sigma} F_{\rho\sigma} g^{\mu\nu} \right), \quad (80)$$

where $F^{\mu\nu}$ is the field strength tensor.