

Some concepts in Special Relativity

Reference frame (RF): a coordinate system for time and space with an origin and orientation

Inertial frame: a non-accelerating RF

we can always
orient the RFs
so that the
relative motion
is in x -direction

A **Galilean transformation** between two reference frames S and S' with relative velocity v in the x -direction is given by:

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$

where (x, y, z, t) and (x', y', z', t') are the coordinates in S and S' , respectively

For a particle with velocity $u = \frac{dx}{dt}$ in S and $u' = \frac{dx'}{dt'}$ in S' we have

$$u' = \frac{dx'}{dt'} = \frac{d}{dt}(x - vt) = \frac{dx}{dt} - v = u - v$$

no need for an
absolute space

Galilean relativity: velocity is relative, without external reference points we can not say if we are moving with constant velocity

Inconsistent with measurements for light where $u = u' = c$

A (special) **Lorentz transformation** or **boost** between the reference frames S and S' is given by

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

where c is the speed of light (in vacuum) and γ is defined as

$$\gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}, \quad \beta \equiv \frac{v}{c}$$

This transformation implies that *time is no longer absolute*

Let us look at the limit where $v \ll c$. From Rottmann p. 115

$$\frac{1}{\sqrt{1 - \beta^2}} = 1 - \frac{1}{2}\beta^2 + \mathcal{O}(\beta^4) \xrightarrow{\beta = \frac{v}{c}} \gamma = 1 + \frac{1}{2}\frac{v^2}{c^2} + \mathcal{O}\left(\frac{v^4}{c^4}\right)$$

With $\frac{v}{c} \ll 1$, $\gamma \simeq 1$ and

$$x' \simeq x - vt, \quad t' \simeq t - \frac{v}{c^2}x \simeq t$$

so Lorentz transformation reduces to Galilean transformation in low velocity limit

What is the relationship between u and u' now?

$$dx' = \gamma(dx - vdt) = \gamma\left(\frac{dx}{dt} - v\right)dt = \gamma(u - v)dt$$

$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right) = \gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)dt = \gamma\left(1 - \frac{uv}{c^2}\right)dt$$

Thus

$$u' = \frac{dx'}{dt'} = \frac{\gamma(u - v)dt}{\gamma(1 - \frac{uv}{c^2})dt} = \frac{u - v}{1 - \frac{uv}{c^2}}$$

the Lorentz
transformation
is the only
linear
transformation
that will do this

So, if $u = c$, then $u' = \frac{c - v}{1 - \frac{cv}{c^2}} = c$

The Lorentz transformations were already known, but Einstein (1905) showed that they followed from assuming that the speed of light is c in all RFs

Rapidity

other common
symbols are y
and w

We can rewrite the Lorentz transformation by defining **rapidity** χ by

$$\cosh \chi = \gamma$$

We then have

$$\sinh \chi = \sqrt{\cosh^2 \chi - 1} = \sqrt{\gamma^2 - 1} = \sqrt{\frac{1}{1 - \beta^2} - 1} = \sqrt{\frac{1 - 1 + \beta^2}{1 - \beta^2}} = \frac{\beta}{\sqrt{1 - \beta^2}} = \beta\gamma$$

and

$$\tanh \chi = \frac{\sinh \chi}{\cosh \chi} = \frac{\beta\gamma}{\gamma} = \beta, \quad \text{so} \quad v = c \tanh \chi$$

Also

$$e^\chi = \cosh \chi + \sinh \chi = \gamma + \beta\gamma = \gamma(1 + \beta) = \frac{1 + \beta}{\sqrt{1 - \beta^2}} = \sqrt{\frac{(1 + \beta)^2}{(1 - \beta)(1 + \beta)}} = \sqrt{\frac{1 + \beta}{1 - \beta}}$$

so we can write rapidity directly in terms of velocity (β)

$$\chi = \frac{1}{2} \ln \frac{1 + \beta}{1 - \beta}$$

Invariant distance

We can extend the (special) Lorentz transformations (boosts) to:

General Lorentz transformations = boost + rotations

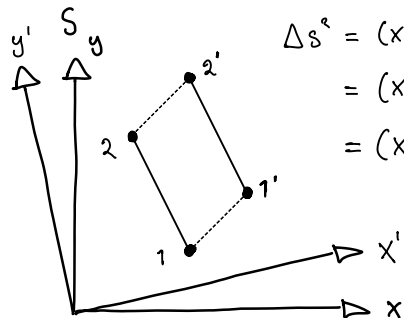
Poincaré transformations = General Lorentz transformations + translations

We want physical laws to be independent of choice of (inertial) reference frames.

for isolated
system we can
eliminate
absolute
velocity and
position by
going to
centre-of-mass
frame

Relative distances Δs and velocities Δv are invariant under rotations (see Lecture 9), translations and Galilean RF transformations

With $\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, $\Delta s' = \Delta s$ under rotation and translation:



The diagram shows two coordinate systems, S (x, y) and S' (x', y'), with S' rotated relative to S. A parallelogram is drawn in the S' frame with vertices labeled 1, 2, 1', and 2'. The vertices 1 and 2 are in the S frame, while 1' and 2' are in the S' frame. Dashed lines connect 1 to 1' and 2 to 2'. The distance between 1' and 2' is labeled $\Delta s'^2$.

$$\begin{aligned} \Delta s'^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= (x_2' - x_1')^2 + (y_2' - y_1')^2 = \Delta s'^2 \\ &= (x_2' - x_1')^2 + (y_2' - y_1')^2 \end{aligned}$$

Since a Galilean RF transformation is in effect a translation $x' = x - vt$ by $-vt$ this will not change a distance

Likewise relative velocities can not change since $\Delta v' = \frac{d\Delta s'}{dt'} = \frac{d\Delta s}{dt} = \Delta v$

Since it mixes
position and
time

This is no longer true for Lorentz transformations!

We can instead define the distance

$$\Delta s^2 = c^2 \Delta t^2 - \Delta \mathbf{r}^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

relies on
linearity of
boost

Then, under a Lorentz transformation,

$$\begin{aligned}
\Delta s'^2 &= c^2 \Delta t'^2 - \Delta x'^2 - \Delta y'^2 - \Delta z'^2 \\
&= c^2 \left[\gamma \left(\Delta t - \frac{v}{c^2} \Delta x \right) \right]^2 - [\gamma (\Delta x - v \Delta t)]^2 - \Delta y^2 - \Delta z^2 \\
&= \gamma^2 \left(c^2 \Delta t^2 - 2v \Delta t \Delta x + \frac{v^2}{c^2} \Delta x^2 \right) - \gamma^2 (\Delta x^2 - 2v \Delta x \Delta t - v^2 \Delta t^2) - \Delta y^2 - \Delta z^2 \\
&= \gamma^2 \left(1 - \frac{v^2}{c^2} \right) c^2 \Delta t^2 + \gamma^2 \left(\frac{v^2}{c^2} - 1 \right) \Delta x^2 - \Delta y^2 - \Delta z^2 \\
&= c^2 \Delta t^2 + \Delta x^2 - \Delta y^2 - \Delta z^2 = \Delta s^2
\end{aligned}$$

So this distance Δs does not change between RFs, it is an **invariant** under Lorentz transformations (and rotations + translations)

In mathematical physics space has geometry, this is described by the **metric tensor** g of the space, which in turn gives the infinitesimal distance (line element) of the space ds

As an example, **Euclidean space** (3D) E_3 with Cartesian coordinates (x, y, z) is a flat space with positive distance

$$ds^2 = dx^2 + dy^2 + dz^2$$

and metric

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We can write the line element as

$$ds^2 = d\mathbf{r}^T g d\mathbf{r} \left(= \sum_{ij} d\mathbf{r}_i g_{ij} d\mathbf{r}_j \right)$$

where $d\mathbf{r} = (dx, dy, dz)$ is a vector of the infinitesimal coordinates

When we instead *define* the line element as the Lorentz invariant

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

we are changing metric (and space) to

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

so that

$$ds^2 = dx g dx$$

with a vector of infinitesimal coordinates $dx = (cdt, dx, dy, dz)$

This space is called (4D) **Minkowski space** M_4

Invariant distance cont.

despite square!

The invariant distance ds^2 or Δs^2 is not necessarily positive

We categorise the distance between two space-time points, **events**, as

$$\Delta s^2 > 0 \quad \textbf{time-like} \quad (\text{we can find RF where } \Delta x = \Delta y = \Delta z = 0)$$

$$\Delta s^2 = 0 \quad \textbf{light-like}$$

$$\Delta s^2 < 0 \quad \textbf{space-like} \quad (\text{we can find RF where } \Delta t = 0)$$

If the two points are on the path of a light signal

$$\Delta s^2 = c^2 \Delta t^2 - \Delta \mathbf{r}^2 = \Delta t^2 \left(c^2 - \frac{\Delta \mathbf{r}^2}{\Delta t^2} \right) = \Delta t^2 (c^2 - c^2) = 0$$

Four-vectors

in order to have
units of length
professionals
often use $c = 1$
to simplify

We can extend our usual space-vectors $\mathbf{r} = (x, y, z)$ to include time. We do this by introducing the time component $x^0 = ct$.

By convention we *index* all of the components as

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

As a vector we write this as the space-time **contravariant four-vector**

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \equiv \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad \text{or} \quad x^T = (x^0, \mathbf{r}) = (ct, \mathbf{r})$$

These form a vector-space, but not an (ordinary) inner product space

If we want to talk about a component of a four-vector we will use greek indices, e.g. x^μ , where $\mu = 0, 1, 2, 3$.

x^μ is often abused to mean x (and we will be abusive)

The length of an ordinary vector $\mathbf{x}$ is given by the inner product (scalar product):

$$\mathbf{x}^2 \equiv \mathbf{x} \cdot \mathbf{x} = \mathbf{x}^T \mathbf{x} = \sum_i x_i x_i = \sum_i x_i^2$$

the x on the left
is a four-vector,
the x on the
right is a
Cartesian
coordinate

The invariant distance can be interpreted as (the square of) the length of a single four-vector x

$$x^2 \equiv c^2 t^2 - x^2 - y^2 - z^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

This can be written using also the **covariant four-vector**

$$x_\mu = (x_0, x_1, x_2, x_3) \equiv (ct, -x, -y, -z)$$

as

$$x^2 = \sum_{\mu=0}^3 x_\mu x^\mu$$

Notice the placement of μ to signal if the vector is contravariant or covariant. The covariant four-vector x_μ lives in the **dual vector space** (just as $\mathbf{x}^T$ does) What is going on is similar to complex vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^n$, where

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^\dagger \mathbf{b}$$

Here $\mathbf{a}$ is technically in the dual space and the difference is seen in the scalar product

Lorentz transformation

A Lorentz transformations (in x -direction) can be written as a matrix multiplication of a space-time four-vector, e.g. $x' = Lx$, where

$$\underbrace{\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix}}_{x'} = \underbrace{\begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_L \underbrace{\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}}_x$$

We can check that

$$x'^0 = ct' = \gamma x^0 - \gamma\beta x^1 = \gamma ct - \gamma\beta x = \gamma c \left(t - \frac{v}{c^2} x \right)$$

so $t' = \gamma \left(t - \frac{v}{c^2} x \right)$, and

$$x'^1 = x' = -\gamma\beta x^0 + \gamma x^1 = -\gamma\beta ct + \gamma x = \gamma(x - vt)$$

while $x'^2 = y' = x^2 = y$ and $x'^3 = z' = x^3 = z$.

We can write this on component form as the sum

$$x'^\mu = \sum_{\nu=0}^3 L^\mu{}_\nu x^\nu$$

where $L^\mu{}_\nu$ is the matrix.

This follows a general rule for four-vector indices:

Four-vector indices that are summed over are repeated, appearing once as a superscript (“upstairs”) and once as a subscript (“downstairs”) in a term. This is called a **contraction**. Indices that are not summed over appear only once in each term, in the same position.

Poincaré transformation

The **Poincaré transformation** has 10 continuous parameters: 3 from rotations (2 from axis + 1 angle), 3 from boosts ($\beta = \frac{v}{c}$), and 4 from the translation

$$x^\mu \rightarrow x'^\mu = x^\mu + a^\mu$$

It forms the **symmetry group** of SR since it leaves $ds^2 = c^2 dt^2 - d\mathbf{r}^2$ invariant

remember
matrix
multiplication
 $\mathbf{b} = A\mathbf{a}$ is
written
 $b_i = \sum_j A_{ij} a_j$

$a^\mu = \begin{pmatrix} a^0 \\ a^1 \\ a^2 \\ a^3 \end{pmatrix}$
using
four-vectors
builds this into
models

In matrix form the Poincaré transformation is

$$x' = Lx + a, \quad \text{or} \quad x'^{\mu} = \sum_{\nu=0}^3 L^{\mu}_{\nu} x^{\nu} + a^{\mu}$$

nor are they
symmetries in
the Standard
Model because
of weak nuclear
force

Some discrete transformations exist that leave ds invariant, but are not part of the Poincaré symmetries,

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (\text{parity}), \quad T = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (\text{time-reversal})$$

Notice that $P^2 = I$, $T^2 = I$, $PT = TP = -I$

Acting on a ordinary vector $P(\mathbf{v}) = -\mathbf{v}$, however, if $\mathbf{w} = \mathbf{a} \times \mathbf{b}$, then $P(\mathbf{w}) = \mathbf{w}$ (**pseudovector**), similarly $P(\mathbf{v} \cdot \mathbf{u}) = \mathbf{v} \cdot \mathbf{u}$ while $P(\mathbf{v} \cdot \mathbf{w}) = -\mathbf{v} \cdot \mathbf{w}$ (**pseudoscalar**)