

Analytical mechanics

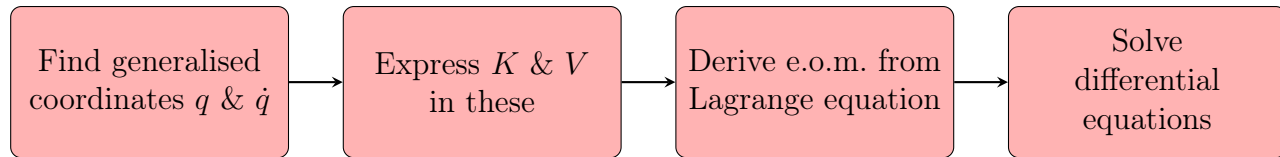
Classical mechanics – as in *not* quantum

may have noticed how many mechanics problems are easier to solve in terms of energy

generalises to fields

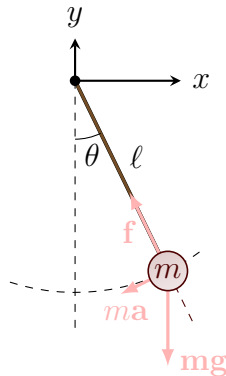
cookbook

- Newton's mechanics: force $\mathbf{F}$ & acceleration $\mathbf{a}$ (vectors)
 $\Rightarrow$ kinetic K & potential V energy (scalars)
- **Lagrange-Hamilton formalism**
 - **Lagrangian:** $L = K - V$
 - **Hamiltonian:** $H = K + V$ [total energy]
- More powerful for complex problems [some loss of intuition]
- Algorithmic:



Example: simple pendulum

Simple 2D pendulum with mass m and length ℓ , with coordinates $\mathbf{r} = (x, y)$



Newton:

complicated set of equations!

$$\begin{aligned} m\mathbf{a} &= m\mathbf{g} + \mathbf{f} \\ |\mathbf{r}| &= \ell \quad (\text{constraint}) \\ \text{or } f(\mathbf{r}) &= |\mathbf{r}| - \ell = 0 \quad (\text{constraint equation}) \end{aligned}$$

Simplify by eliminating one coordinate using constraint

$$\mathbf{r} = (\ell \sin \theta, -\ell \cos \theta) = \mathbf{r}(\theta)$$

dot means total time derivative, i.e. $\dot{x} = \frac{dx}{dt}$

We now have only one coordinate θ and the velocity is given through $\dot{\theta}$

$$\dot{\mathbf{r}} = (\dot{x}, \dot{y}) = (\ell \cos \theta \dot{\theta}, \ell \sin \theta \dot{\theta})$$

notice here how
the kinetic term
keeps its form
 $\frac{1}{2}M\dot{\theta}^2$ with an
effective mass
 $M = m\ell^2$

The kinetic and potential energy are also functions of θ and $\dot{\theta}$

$$\begin{aligned} K &= \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\ell^2 \cos^2 \theta \dot{\theta}^2 + \ell^2 \sin^2 \theta \dot{\theta}^2) = \frac{1}{2}m\ell^2 \dot{\theta}^2 \\ V &= mgy = -mg\ell \cos \theta \end{aligned}$$

The Lagrangian is

$$L = K - V = \frac{1}{2}m\ell^2 \dot{\theta}^2 + mg\ell \cos \theta$$

and the Euler-Lagrange equation gives

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = m\ell^2 \ddot{\theta} + mg\ell \sin \theta = 0$$

which is the differential equation we need to solve

Generalised coordinates

Generalised coordinates = remaining coordinates after constraints imposed

of generalised coordinates = degrees of freedom (d.o.f.)

In a 3D system with N bodies with Cartesian coordinates $\mathbf{r}_i$, $i = 1, 2, \dots, N$, and with M **holonomic constraints** (velocity independent)

$$f_j(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, t) = 0, \quad j = 1, 2, \dots, M$$

the d.o.f. are

$$d = 3N - M$$

not a vector
space!

We write the **generalised coordinates** as q_k , $k = 1, 2, \dots, d$, and we write the Cartesian coordinates as

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \dots, q_d, t) \quad \text{or just} \quad \mathbf{r}_i = \mathbf{r}_i(q, t), \quad q = \{q_1, q_2, \dots, q_d\}$$

We can now find K and V in terms of q and $\dot{q}$

many examples
in Sec. 1.1 of
Leinaas' book

The choice of q is a matter of good taste!

Configuration space

manifold: space
locally
Euclidean, e.g.
sphere surface

The generalised coordinates $q = \{q_1, \dots, q_d\}$ form a d -dimensional space (**manifold**) called **configuration space**

With N particles the position of all particles is given by a $3N$ -dim vector

$$\mathbf{R} = \mathbf{R}(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N)$$

t appears here if
the constraint is
time-dependent

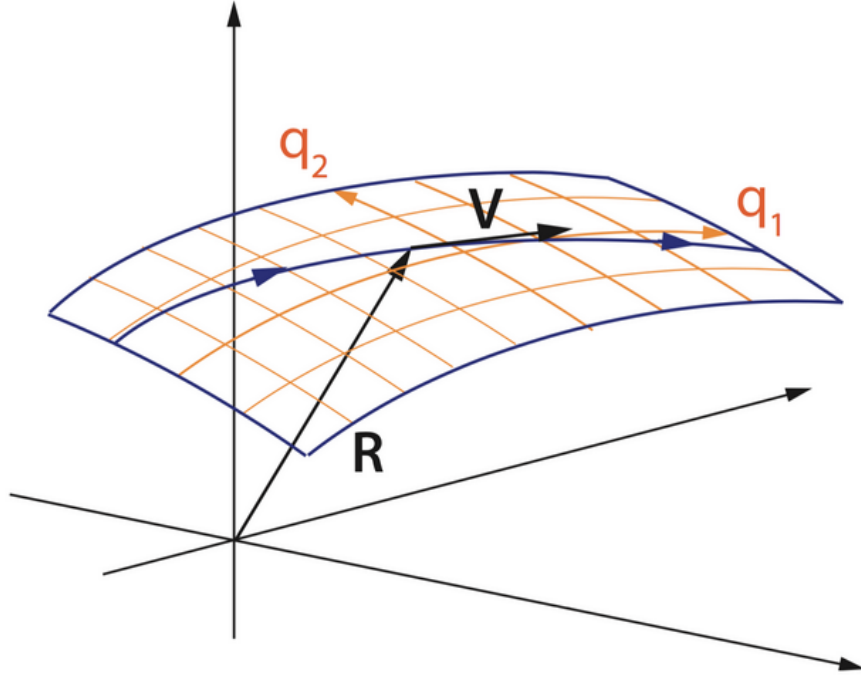
In terms of the generalised coordinates

$$\mathbf{R} = \mathbf{R}(q_1, \dots, q_d; t)$$

not in general a
vector space

This is a hypersurface of dimension $d = 3N - M$ in the $3N$ -dim space with q as a coordinate system

Example with $N = 1$ particle, $M = 1$ constraint (must lie on surface), and $d = 3 - 1 = 2$ generalised coordinates q_1 and q_2



If constraints are time-independent the surface is fixed and time evolution of $\mathbf{R}$ is a curve on the surface

$$\mathbf{R}(t) = \mathbf{R}(q(t))$$

If constraints are time dependent and surface moves, then

$$\mathbf{R}(t) = \mathbf{R}(q(t), t)$$