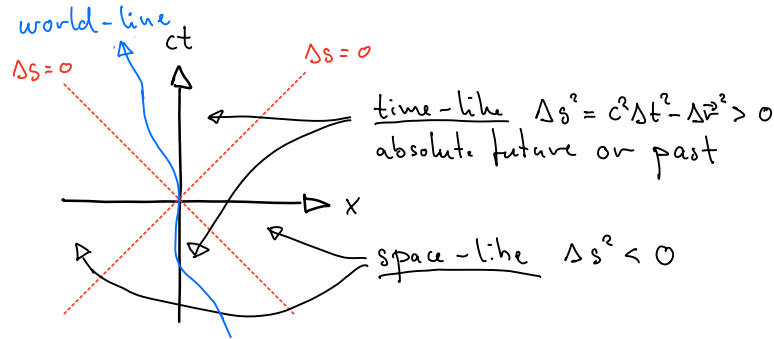


Minkowski diagrams

We often draw diagrams of 2D $(x^0, x^1) = (ct, x)$ or 3D $(x^0, x^1, x^2) = (ct, x, y)$ sub-spaces of Minkowski space called **Minkowski diagrams**

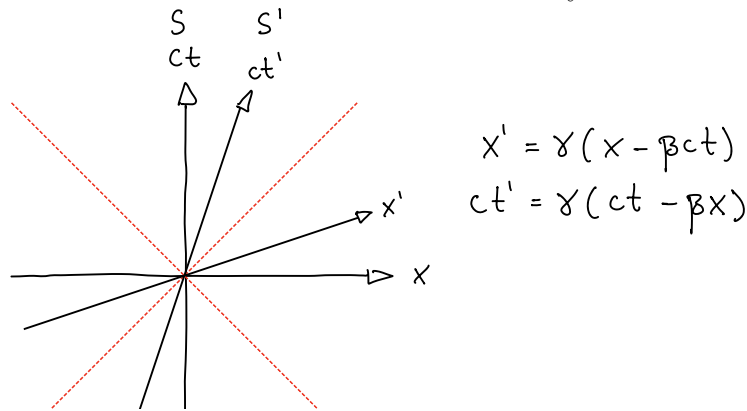


The line (blue) that an object follows is called its **world-line**

The ct axis is the world-line of an observer at rest in the RF

Nothing can propagate faster than light (red line), i.e. with an angle less than 45°

We can draw two RFs S and S' related by a boost in the same diagram



Points on the x -axis (x' -axis) or a parallel line are simultaneous in S (in S')

These are not the same points! Simultaneous events are RF dependent!

Lorentz transformations can bring space-like events into past or future

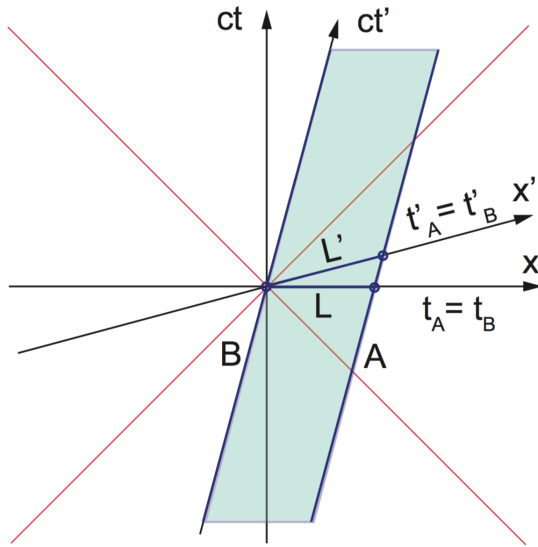
Length contraction

A rod at rest in RF S' has length $L' = x'_B - x'_A$, where A and B are the endpoints of the rod along the x' -axis

The rod moves with velocity v in the x -direction with respect to RF S , called the **laboratory frame**

The relationship between the coordinates in the two RFs is

$$\begin{aligned} x'_A &= \gamma(x_A - vt_A), & t'_A &= \gamma\left(t_A - \frac{v}{c^2}x_A\right) \\ x'_B &= \gamma(x_B - vt_B), & t'_B &= \gamma\left(t_B - \frac{v}{c^2}x_B\right) \end{aligned}$$



In RF S we must specify also the time we measure x_A and x_B as the rod is moving
We use the following (natural?) definition:

The **proper distance** of a moving body measured in an inertial RF S
is the space distance between its endpoints measured at equal times in S

So, we must use $t_A = t_B$ to measure length $L = x_B - x_A$. Then

$$L' = x'_B - x'_A = \gamma[(x_B - x_A) - v(t_B - t_A)] = \gamma(x_B - x_A) = \gamma L$$

$1 \leq \gamma < \infty$ We have **length contraction** because $L = \frac{1}{\gamma} L' \leq L'$

Time dilation

A clock is at rest in the origin of RF S' and moving with velocity v in the x -direction
with respect to RF S which has another clock

Assume both clocks are started with $t_0 = t'_0 = 0$ when $x_0 = x'_0 = 0$

we will see why
we use τ for this
time below

The next “tick” of the clock in S' happens at $(x', t') = (0, \tau)$ in RF S' , and at (x, t)
is S

The relationship between the coordinates are

$$\begin{aligned} t' &= \gamma \left(t - \frac{v}{c^2} x \right) \\ x' &= \gamma (x - vt) \end{aligned}$$

because S is
moving with
velocity $-v$
relative to S'

To find time in S we must solve for t (and x). Either see that it is obvious that

$$\begin{aligned} t &= \gamma \left(t' + \frac{v}{c^2} x' \right) \\ x &= \gamma (x' + vt') \end{aligned}$$

or show this (exercise)

We then find that the “tick” takes the following time observed in S

$$t = \gamma \left(\tau + \frac{v}{c^2} \cdot 0 \right) = \gamma \tau$$

So $t = \gamma\tau \geq \tau$, which is **time dilation** (lengthening)

It is important that the situation is *symmetric*. Seen from S' the clock in S is slow since S is moving with velocity $-v$

If a “tick” in S takes time τ then the time in S' is

$$t' = \gamma \left(\tau - \frac{v}{c^2} x \right) = \gamma\tau \quad (\text{clock is at } x = 0)$$

So both RFs are observing the other clock as slower

we need a less
confusing time...

proper length
is the length in
the rest frame
also known as
**instantaneous
comoving RF**

Proper time

The **rest frame** RF is the RF that follows a body with constant velocity

The time in the rest frame RF is called the **proper time** τ .

This can be extended to an accelerating body using the **instantaneous inertial RF** (IIRF), the rest frame corresponding to the velocity $v(t)$ at any particular moment t . For an accelerating body the IIRF is continuously changing, however, the proper time is still the time measured by a clock moving *with* the body

From time dilation we have at any instance

$$\tau = \frac{1}{\gamma} t = \sqrt{1 - \frac{v^2}{c^2}} t$$

and expressed as infinitesimals

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}} dt$$

for the time $d\tau$ measured in the IIRF compared to dt in some other inertial RF.

The proper time between events A and B is then

$$\tau_{AB} = \int_{t_A}^{t_B} \sqrt{1 - \frac{v(t)^2}{c^2}} dt \quad (1)$$

This time is the time measured for a clock on the body, while $v(t)$ is the speed of the body in some other RF. Is this independent on the choice of the other RF? Yes!

The invariant distance can be expressed as

$$ds^2 = c^2 dt^2 - d\mathbf{r}^2 = c^2 dt^2 - v^2 dt^2 = \left(1 - \frac{v^2}{c^2}\right) c^2 dt^2 = c^2 d\tau^2$$

this is the
second invariant
we have found

ds^2 invariant under Lorentz transformations $\Rightarrow d\tau^2$ is invariant

However, it is not path independent: highest average v^2 has lowest τ

Twin paradox

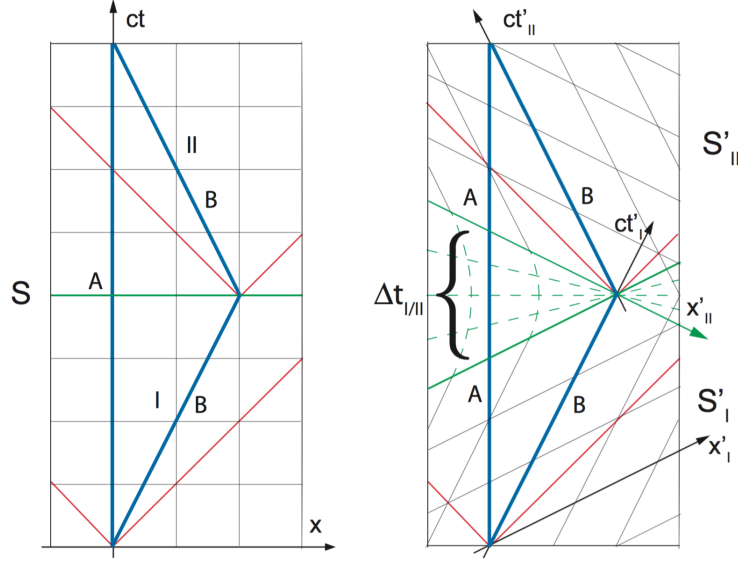
We have two twins Alice (A) and Bob (B)

B leaves on a space ship travelling for years to a space station and back with constant speed v near the speed of light

When B returns, B has aged less than A because of time dilation seen from A

this is off course
silly, formula (1)
is correct

The “paradox” is: We can object by saying that observed from B , A is travelling away at speed $-v$, and since the γ factor in the time-dilation formula depends on $(-v)^2 = v^2$, B could claim A is younger by the same amount



For A in RF S the proper time the trip took is denoted τ_A .
 A can calculate the proper time for B , τ_B by the time dilation formula:

$$\tau_B = \underbrace{\frac{1}{\gamma} \frac{\tau_A}{2}}_{\text{there}} + \underbrace{\frac{1}{\gamma} \frac{\tau_A}{2}}_{\text{back again}} = \frac{1}{\gamma} \tau_A < \tau_A \quad (2)$$

B has to be careful, he is RF S'_I going out and in S'_{II} going back
 By the time dilation formula, on the way there the clock in S should show

$$\tau_A^{\text{there}} = \frac{1}{\gamma} \frac{\tau_B}{2}$$

and the same going back

The confusion now arises because seen from B $\tau_A^{\text{there}} \neq \frac{\tau_A}{2}$, in fact $\tau_A^{\text{there}} < \frac{\tau_A}{2}$.
 Because of the switch of RFs, the beginning of the return journey (in S'_{II}) is simultaneous with a time that is later in S by $\Delta t_{I/II}$, and the total time should be

$$\tau_A = 2\tau_A^{\text{there}} + \Delta t_{I/II} = \frac{1}{\gamma} \tau_B + \Delta t_{I/II}$$

By comparison to (2) this extra time must be

$$\Delta t_{I/II} = \tau_A - \frac{1}{\gamma} \tau_B = \tau_A - \frac{1}{\gamma^2} \tau_A = \left(1 - \frac{1}{\gamma^2}\right) \tau_A = \left(1 - 1 + \frac{v^2}{c^2}\right) \tau_A = \frac{v^2}{c^2} \tau_A$$

The problem here is caused by ignoring acceleration and the continuous change of RF for the proper time