

Einstein's summation convention

Einstein: "I have made a great discovery in mathematics; I have suppressed the summation sign every time that the summation must be made over an index which occurs twice..."

When an index appears twice in a single term it is a **summation index**, and implies summation over all the values of the index.

An index that is not summed over is a **free index** and should appear once and once only in every term in the equation

A summation index is also called a **dummy index** since it can be replaced with any other (unused) index without changing the meaning of the expression

Examples:

$$y_i = A_{ij}x_j \quad \text{means} \quad y_i = \sum_j A_{ij}x_j$$

$$y^\mu = L^\mu_\nu x^\nu + a^\mu \quad \text{means} \quad y^\mu = \sum_{\nu=0}^3 L^\mu_\nu x^\nu + a^\mu$$

Raising and lowering indices

We wrote the infinitesimal line element ds using the metric tensor g as

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = dx^T g dx, \quad \text{where} \quad g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad dx = \begin{bmatrix} cdt \\ dx \\ dy \\ dz \end{bmatrix}$$

so, no free indices

or in component form using Einstein's summation convention

$$ds^2 = dx^\mu g_{\mu\nu} dx^\nu = dx^\mu dx_\mu$$

where $dx_\mu = (ct, -dx, -dy, -dz)$ is the covariant four-vector

So we can use the metric tensor $g_{\mu\nu}$ to **lower** and **raise** four-vector indices

$$x_\mu = g_{\mu\nu} x^\nu \quad \text{and} \quad x^\mu = g^{\mu\nu} x_\nu$$

thus also $g^T = g$

Note that since g is diagonal $g_{\mu\nu}$ is symmetric in indices, i.e. $g_{\mu\nu} = g_{\nu\mu}$
 $g^{-1} = g$ since $gg^{-1} = gg = I$. In terms of components we write this as

$$g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho \quad \text{so that} \quad x_\mu = g_{\mu\nu} g^{\nu\rho} x_\rho$$

where δ_μ^ρ is the **Kronecker delta** for four-vector indices

General four-vectors

We can generalise from space-time x^μ to a **general four-vector** A^μ which:

- 1) Has four components A_μ , $\mu = 0, 1, 2, 3$
- 2) Its indices can be raised and lowered with $g^{\mu\nu}$ and $g_{\mu\nu}$

$$A_\mu = g_{\mu\nu} A^\nu \quad \text{and} \quad A^\mu = g^{\mu\nu} A_\nu$$

3) Its components transform like x^μ under Lorentz transformations

$$A^\mu \rightarrow A'^\mu = L^\mu_\nu A^\nu$$

We also call these **Lorentz vectors**

We can define the **scalar product** for two four-vectors A^μ and B^μ in analogue to the invariant distance ds^2 as

$$AB \equiv g_{\mu\nu} A^\mu B^\nu = A^\mu B_\mu = A_\mu B^\mu$$

which we call the **contraction** of A and B

need to know
more properties
of the Lorentz
transformation
first

The contraction of two Lorentz vectors (four-vectors) is always invariant under Lorentz transformations (proof later)

The scalar product $A^2 = A_\mu A^\mu$ is **indefinite** (not always positive)

We say that for

$$A^2 > 0 \quad A \text{ is time-like}$$

$$A^2 = 0 \quad A \text{ is light-like}$$

$$A^2 < 0 \quad A \text{ is space-like}$$

Tensors

Tensors are multi-dimensional arrays, e.g. vectors or matrices

Example: a matrix A , a vector $\mathbf{y}$, x_μ , L^μ_ν , $g_{\mu\nu}$

The number of indices is the **rank** of the tensor

A	rank-0 scalar
B^μ	rank-1 (four)-vector
$C^{\mu\nu}$	rank-2 tensor
$D^{\mu\nu\rho}$	rank-3 tensor

With two four-vectors A^μ and B^μ we can form the **tensor product** $C^{\mu\nu} = A^\mu B^\nu$

Under Lorentz transformations this transforms as

$$C^{\mu\nu} \rightarrow C'^{\mu\nu} = L^\mu_\rho L^\nu_\sigma C^{\rho\sigma}$$

does not need to
be made from a
tensor product

A **Lorentz tensor (four-tensor)** is a tensor where each index transforms according to the Lorentz transformation

We also have $C^\mu_\nu = g_{\nu\rho} C^{\mu\rho}$, $C^\mu_\nu = g_{\mu\rho} C^{\rho\nu}$ and $C_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} C^{\rho\sigma}$

Note that A^μ_μ is a scalar, and $B^{\mu\nu}_\nu$ is a vector

Here $A^\mu_\mu = \sum_{\mu=0}^3 A^\mu_\mu$ is the sum of the diagonal entries of A , or $\text{tr } A$

Lorentz transformations

We can write Lorentz transformation as

$$x'^\mu = L^\mu_\nu x^\nu$$

In order to get the needed invariance $ds'^2 = ds^2$ we must have

$$\begin{aligned} ds'^2 &= g_{\mu\nu} dx'^\mu dx'^\nu = g_{\mu\nu} L^\mu_\rho dx^\rho L^\nu_\sigma dx^\sigma \\ &= g_{\mu\nu} L^\mu_\rho L^\nu_\sigma dx^\rho dx^\sigma \\ ds^2 &= g_{\rho\sigma} dx^\rho dx^\sigma \end{aligned}$$

Thus

$$g_{\mu\nu} L^\mu_\rho L^\nu_\sigma = g_{\rho\sigma} \quad (1)$$

but not
sufficient, e.g.
also g , P , and T
fulfils this

This is a necessary condition on L for it to be a Lorentz transformation
In matrix form this is

$$L^T g L = g \quad (2)$$

We need to find the Lorentz transformation for a covariant four-vector A_μ
We have

$$A'_\mu = g_{\mu\nu} A'^\nu = g_{\mu\nu} L^\nu_\rho A^\rho = g_{\mu\nu} L^\nu_\rho g^{\rho\sigma} A_\sigma$$

so the Lorentz transformation must be

$$L_\mu^\sigma = g_{\mu\nu} L^\nu_\rho g^{\rho\sigma}$$

giving $A'_\mu = L_\mu^\sigma A_\sigma$

for boost in

The matrix of L_μ^σ is $\tilde{L}$ given by

$$\tilde{L} = g L g^{-1} = (L^T)^{-1} g g^{-1} = (L^T)^{-1}$$

$$\begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

since from (2) $L^T g L = g \Rightarrow g L = (L^T)^{-1} g$

We can now finally prove that all contractions are invariant under Lorentz transformations, i.e. $A'B' = AB$:

$$A'B' = A'^\mu B'_\mu = L^\mu_\nu A^\nu L_\mu^\rho B_\rho = L^\mu_\nu g_{\mu\sigma} L^\sigma_\tau g^{\tau\rho} A^\nu B_\rho \stackrel{(1)}{=} g_{\nu\tau} g^{\tau\rho} A^\nu B_\rho = \delta_\nu^\rho A^\nu B_\rho = AB$$