

Vector and tensor fields

We can also have **four-vector** and **tensor fields**

$$\begin{aligned}\phi(x) & \text{ scalar field } (x \text{ is a four-vector}) \\ A^\mu(x) & \text{ vector field} \\ F^{\mu\nu}(x) & \text{ rank-2 tensor field}\end{aligned}$$

These transform under Lorentz transformations in the following way

$$\begin{aligned}\phi(x) & \rightarrow \phi'(x') = \phi(x) \\ A^\mu(x) & \rightarrow A'^\mu(x') = L^\mu{}_\nu A^\nu(x) \\ F^{\mu\sigma}(x) & \rightarrow F'^{\mu\nu}(x') = L^\mu{}_\rho L^\nu{}_\sigma F^{\rho\sigma}(x)\end{aligned}$$

Note that both the field and x transforms, so it is the field for the same point

we need to
prove this is a
four-vector

Example: The **four-potential**

$$A^\mu(x) = \left(\frac{1}{c} \phi(x), \mathbf{A}(x) \right)$$

where $\phi(x) = \phi(\mathbf{r}, t)$ and $\mathbf{A}(x) = \mathbf{A}(\mathbf{r}, t)$ are the scalar and vector potentials.

Differentiation with four-vectors

We define the **four-derivative** or **four-gradient** as

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$$

note the lack of
minus sign and
position of index

or in components

$$\partial_\mu = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right) = \left(\frac{1}{c} \frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right)$$

Here we used $x^0 = ct$, so that

$$\frac{\partial}{\partial x^0} = \frac{\partial t}{\partial x^0} \frac{\partial}{\partial t} = \frac{1}{c} \frac{\partial}{\partial t}$$

So ∂_μ has four components. Is it a four-vector? Yes!

Proof: We can use the chain rule to find derivative w.r.t. the Lorentz transformed $x'^\mu = L^\mu{}_\nu x^\nu$, ∂'_μ ,

$$\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^\mu} \partial_\nu \quad (\text{a sum over } \nu)$$

To find the derivative we need the inverse of the Lorentz transformation

Lecture 13 We saw that $g_{\mu\nu} L^\mu{}_\rho L^\nu{}_\sigma = g_{\rho\sigma}$. Thus

$$g_{\mu\nu} L^\mu{}_\rho L^\nu{}_\sigma x^\sigma = g_{\mu\nu} L^\mu{}_\rho x'^\nu = x_\rho$$

and then by raising ρ and lowering ν we have

$$\rho \rightarrow \nu, \nu \rightarrow \mu \quad x^\rho = L_\nu^\rho x'^\nu \quad \text{or} \quad x^\nu = L_\mu^\nu x'^\mu$$

we can show as the inverse Lorentz transformation, so

$$\text{that} \\ L^\mu_\nu L^\nu_\sigma = \delta^\mu_\sigma \quad \frac{\partial x^\nu}{\partial x'^\mu} = L_\mu^\nu$$

Then $\partial'_\mu = L_\mu^\nu \partial_\nu$, which shows that ∂_μ is a covariant four-vector

As a consequence $\partial_\mu \phi(x)$ is a four-vector field since under a Lorentz transformation

$$\partial_\mu \phi(x) \rightarrow \partial'_\mu \phi'(x') = L_\mu^\nu \partial_\nu \phi(x)$$

We can easily extend this to for example $\partial_\mu A^\mu(x)$ which is a scalar or $\partial^\mu A^\nu(x)$ which is a rank-2 tensor field

Example: From the four-potential we can define the **electromagnetic field strength tensor**

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$$

Raising the index on the four-derivative gives

$$\partial^\mu = g^{\mu\nu} \partial_\nu = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) = \left(\frac{1}{c} \frac{\partial}{\partial t}, -\nabla \right)$$

We can also form the **d'Alembertian**

$$(\square =) \partial_\mu \partial^\mu = \left(\frac{\partial}{\partial x^0} \right)^2 - \left(\frac{\partial}{\partial x^1} \right)^2 - \left(\frac{\partial}{\partial x^2} \right)^2 - \left(\frac{\partial}{\partial x^3} \right)^2 = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

which is a scalar differential operator

this corresponds
to the
Schrödinger
equation for
non-relativistic
QM

Example: The relativistic quantum mechanical equation for a scalar (spin-0) particle of mass m is the **Klein-Gordon equation**

$$\left(\partial_\mu \partial^\mu + \left(\frac{mc}{\hbar} \right)^2 \right) \phi(x) = 0$$

when $\phi(x)$ is the particle wave-function.

For a spin- $\frac{1}{2}$ particle the correct relativistic equation is the **Dirac-equation**

$$(i\hbar \gamma^\mu \partial_\mu - mc) \psi(x) = 0$$

where $\psi(x)$ is the wave-function, and γ^μ is a four-vector (of 4×4 matrices)

Levi–Civita symbol

tensor-like but
not a tensor

The **Levi–Civita symbol** in two dimensions is

$$\epsilon_{ij} = \begin{cases} 1 & \text{if } (i, j) = (1, 2) \\ -1 & \text{if } (i, j) = (2, 1) \\ 0 & \text{if } i = j \end{cases}$$

This is anti-symmetric, $\epsilon_{ij} = -\epsilon_{ji}$, and can be written as a matrix

$$\epsilon = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Properties:

$$\begin{aligned} \epsilon_{ij}\epsilon_{mn} &= \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} \\ \epsilon_{ij}\epsilon_{in} &= \delta_{jn} \\ \epsilon_{ij}\epsilon_{ij} &= 2 \end{aligned}$$

In three dimensions

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (i, j, k) \text{ is } (1, 2, 3), (2, 3, 1), (3, 2, 1) \\ -1 & \text{if } (i, j, k) \text{ is } (3, 2, 1), (2, 1, 3), (1, 3, 2) \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases}$$

This is anti-symmetric under the exchange (permutation) of any two indices:

$$\epsilon_{ijk} = -\epsilon_{ikj} = \epsilon_{kij}$$

which can be used to define the Levi–Civita symbols in higher dimensions

Properties:

$$\begin{aligned} \epsilon_{ijk}\epsilon_{ilm} &= \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl} \\ \epsilon_{ijk}\epsilon_{ijl} &= 2\delta_{kl} \\ \epsilon_{ijk}\epsilon_{ijk} &= 6 \\ \mathbf{a} \times \mathbf{b} &= \epsilon_{ijk}\hat{\mathbf{e}}_i a_j b_k \quad \text{or} \quad (\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k \\ \det(A) &= \epsilon_{ijk} a_{1i} a_{2j} a_{3k} \quad \text{for a } 3 \times 3 \text{ matrix } A \text{ with elements } a_{ij} \end{aligned}$$

Example: $\mathbf{B} = \nabla \times \mathbf{A}$ can be written $\mathbf{B} = \epsilon_{ijk}\hat{\mathbf{e}}_i \frac{\partial}{\partial x_j} A_k$ or $B_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} A_k$.

Why is ϵ_{ijk} not a tensor? Ordinary tensors transform as the basis transforms, e.g. under a parity transformation vectors change sign, ϵ_{ijk} does not. It is a **pseudo-tensor** (and so is $\mathbf{a} \times \mathbf{b}$!)

In Minkowski space we define the Levi–Civita symbol

$$\epsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } (\mu, \nu, \rho, \sigma) = (0, 1, 2, 3) \text{ or even } \# \text{ permutations} \\ -1 & \text{for odd permutations} \\ 0 & \text{if any indices are identical} \end{cases}$$

Warning! With

$$\epsilon_{\mu\nu\rho\sigma} = g_{\mu\alpha} g_{\nu\beta} g_{\rho\gamma} g_{\sigma\delta} \epsilon^{\alpha\beta\gamma\delta}$$

we have $\epsilon_{0123} = g_{0\alpha} g_{1\beta} g_{2\gamma} g_{3\delta} \epsilon^{\alpha\beta\gamma\delta} = 1 \cdot (-1) \cdot (-1) \cdot (-1) \cdot \epsilon^{0123} = -1$

Four-velocity

The path of a particle in some RF can be described by the relativistic four-vector

$$x^\mu(t) = (ct, \mathbf{r}(t))$$

However,

$$\frac{dx^\mu}{dt} = \left(c, \frac{d\mathbf{r}(t)}{dt} \right)$$

is not a four-vector/Lorentz vector

see Leinaas' book for a specific demonstration

This does not transform with the Lorentz transformation when changing RFs, since t does not transform in the right way

We define **four-velocity** $U^\mu = (U^0, \mathbf{U})$ for a massive particle instead using the proper time τ of the particle

$$U^\mu \equiv \frac{dx^\mu}{d\tau}$$

We argued earlier that since

$$ds^2 = dx_\mu dx^\mu = c^2 dt^2 - d\mathbf{r}^2 = c^2 d\tau^2$$

and ds is a Lorentz invariant, so must $d\tau$ be

only x^μ transforms

Since dx^μ is a four-vector, U^μ must be a four-vector

We can find an explicit expression for U^μ

$$U^\mu = \frac{dx^\mu}{d\tau} = \frac{d}{d\tau} (ct, \mathbf{r}) = \frac{dt}{d\tau} \frac{d}{dt} (ct, \mathbf{r}) = \gamma(c, \mathbf{v})$$

central

where

relationship between proper time τ and time t in another inertial RF

$$d\tau = \frac{1}{c} \sqrt{c^2 dt^2 - d\mathbf{r}^2} = \frac{1}{c} \sqrt{c^2 - \left(\frac{d\mathbf{r}}{dt} \right)^2} dt = \sqrt{1 - \frac{v^2}{c^2}} dt = \frac{1}{\gamma} dt \quad \text{so} \quad \frac{dt}{d\tau} = \gamma$$

We can then find the contraction

$$U^2 = U^\mu U_\mu = \gamma^2 (c^2 - \mathbf{v}^2) = \frac{c^2 - v^2}{1 - \frac{v^2}{c^2}} = c^2$$

So four-velocity squared U^2 is a Lorentz invariant and time-like (we can find RF where $\mathbf{U} = 0$)