

## Four-acceleration

In GR we define We define **four-acceleration**  $A^\mu = (A^0, \mathbf{A})$  by

$$A^\mu = \frac{dU^\mu}{d\tau} + \Gamma^\mu_{\rho\sigma} U^\rho U^\sigma \quad A^\mu \equiv \frac{dU^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2}$$

where  $\Gamma$  describes geometry This is automatically a Lorentz vector since it is defined from a Lorentz vector  $dU^\mu$  and an invariant  $d\tau$ .

acceleration in some RF What is the relation to  $\mathbf{a} = \frac{d\mathbf{v}}{dt}$  ?

$$A^\mu = \frac{dt}{d\tau} \frac{d}{dt} U^\mu = \gamma \left( \frac{dU^0}{dt}, \frac{d\mathbf{U}}{dt} \right)$$

Here

$$\begin{aligned} U^0 = c\gamma \quad \text{so} \quad \frac{dU^0}{dt} &= c \frac{d\gamma}{dt} \\ \mathbf{U} = \gamma \mathbf{v} \quad \text{so} \quad \frac{d\mathbf{U}}{dt} &= \gamma \frac{d\mathbf{v}}{dt} + \frac{d\gamma}{dt} \mathbf{v} = \gamma \mathbf{a} + \frac{d\gamma}{dt} \mathbf{v} \end{aligned}$$

We need to find the time derivative of  $\gamma$  :

$$\frac{d}{dt} \mathbf{v}^2 = 2\mathbf{v} \cdot \frac{d\mathbf{v}}{dt} \quad \frac{d\gamma}{dt} = \frac{d}{dt} \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} = \left( 1 - \frac{v^2}{c^2} \right)^{-3/2} \left( -\frac{1}{2} \right) \left( -\frac{2\mathbf{v}}{c^2} \right) \cdot \frac{d\mathbf{v}}{dt} = \gamma^3 \frac{1}{c^2} \mathbf{v} \cdot \mathbf{a}$$

This gives

$$\begin{aligned} A^0 &= \gamma c \frac{d\gamma}{dt} = \gamma^4 \frac{\mathbf{v} \cdot \mathbf{a}}{c} \\ \mathbf{A} &= \gamma^2 \mathbf{a} + \gamma \frac{d\gamma}{dt} \mathbf{v} = \underbrace{\gamma^2 \mathbf{a}}_{\text{from time dilation}} + \underbrace{\gamma^4 \frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \mathbf{v}}_{\text{from time derivative of dilation}} \end{aligned}$$

or

$$A^\mu = \gamma^2 \left( \gamma^2 \frac{\mathbf{v} \cdot \mathbf{a}}{c}, \mathbf{a} + \gamma^2 \frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \mathbf{v} \right)$$

Note

$$A^\mu U_\mu = \frac{dU^\mu}{d\tau} U_\mu = \frac{1}{2} \frac{d}{d\tau} (U^\mu U_\mu) = \frac{1}{2} \frac{d}{d\tau} c^2 = 0$$

so  $A^\mu$  and  $U_\mu$  are orthogonal

The Lorentz invariant  $A^2 = A_\mu A^\mu$  is

$$\begin{aligned} A^2 &= A_\mu A^\mu = (A^0)^2 - \mathbf{A}^2 \\ &= \gamma^8 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} - \gamma^4 \mathbf{a}^2 - 2\gamma^6 \frac{\mathbf{v} \cdot \mathbf{a}}{c^2} \mathbf{a} \cdot \mathbf{v} - \gamma^8 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^4} \mathbf{v}^2 \\ &= \gamma^8 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} \left( 1 - \frac{\mathbf{v}^2}{c^2} \right) - \gamma^4 \mathbf{a}^2 - 2\gamma^6 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} \\ &= -\gamma^4 \mathbf{a}^2 - \gamma^6 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^4} \end{aligned}$$

Since  $A^2 < 0$ ,  $A^\mu$  is space-like and we can find a RF where  $A^0 = 0$

This happens for  $\mathbf{v} \cdot \mathbf{a} = 0$ , either  $\mathbf{v}$  and  $\mathbf{a}$  are orthogonal (Q: what is this?) or  $\mathbf{v} = 0$ , i.e. we are in the instantaneous inertial RF (IIRF)

The acceleration in the IIRF is the **proper acceleration**  $\mathbf{a}_0(\tau)$ , and the four-acceleration there is

$$A^\mu(\tau) = (0, \mathbf{a}_0(\tau))$$

Using the Lorentz invariant  $A^2$  we have

$$\mathbf{a}_0^2 = \gamma^4 \mathbf{a}^2 + \gamma^6 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2}$$

where the left-hand side is in the IIRF and the right-hand side any other inertial RF  
This shows that  $a_0 \geq a$  for  $\mathbf{a}$  in any other RF

think satellite  
orbiting Earth

*Example:* Motion in circular orbit  $\mathbf{v} \cdot \mathbf{a} = 0$ , then  $a_0 = \gamma^2 a$

*Example:* Linear acceleration  $\mathbf{v} \cdot \mathbf{a} = va$ . Then

$$\begin{aligned} a_0^2 &= \gamma^4 \left( a^2 + \gamma^2 \frac{v^2}{c^2} a^2 \right) = \gamma^6 \left( \frac{1}{\gamma^2} + \beta^2 \right) a^2 \\ &= \gamma^6 (1 - \beta^2 + \beta^2) a^2 = \gamma^6 a^2 \end{aligned}$$

so  $a_0 = \gamma^3 a$

## Four-momentum

this is  
automatically a  
Lorentz vector  
since  $U^\mu$  is

We form the relativistic **four-momentum**  $p^\mu = (p^0, \mathbf{p})$  for a massive particle by

$$p^\mu \equiv mU^\mu = (\gamma mc, \gamma m\mathbf{v}) = \left( \frac{E}{c}, \mathbf{p} \right)$$

We now call

$$E \equiv \gamma mc^2 \quad \text{and} \quad \mathbf{p} \equiv \gamma m\mathbf{v}$$

the **relativistic energy** and **momentum**

In the low velocity limit  $\gamma \simeq 1$  so  $\mathbf{p} \simeq m\mathbf{v}$  and

$$E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} = mc^2 \left( 1 + \frac{1}{2} \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} + \dots \right) = \underbrace{mc^2}_{E_0} + \underbrace{\frac{1}{2}mv^2}_K + \dots$$

The energy  $E_0 = mc^2$  is the **rest energy** of a particle.

We can form the Lorentz invariant

$$p^2 = m^2 U^2 \qquad p^2 = p_\mu p^\mu = \frac{E^2}{c^2} - \mathbf{p}^2 = \gamma^2 m^2 c^2 - \gamma^2 m^2 \mathbf{v}^2 = \gamma^2 \left( 1 - \frac{v^2}{c^2} \right) m^2 c^2 = m^2 c^2$$

This gives the relativistic **energy-momentum relationship**

$$E^2 = p^2 c^2 + m^2 c^4$$

This expression has a sensible limit when  $m \rightarrow 0$ ,

$$E = pc \quad \text{where} \quad p = |\mathbf{p}|$$

and then

$$\frac{\mathbf{p}}{E} = \frac{\gamma m \mathbf{v}}{\gamma m c^2} = \frac{\mathbf{v}}{c^2} \Rightarrow \mathbf{v} = \frac{\mathbf{p}}{E} c^2 \xrightarrow{m \rightarrow 0} \frac{\mathbf{p}}{p} c = c \hat{\mathbf{p}}$$

so the magnitude of the velocity of a massless particle is  $c$ , and the four-momentum

$$p^\mu = (\gamma m c, \gamma m \mathbf{v}) = \frac{E}{c} \left(1, \frac{\mathbf{p}}{E} c\right) \xrightarrow{m \rightarrow 0} p(1, \hat{\mathbf{p}}) = (p, \mathbf{p})$$

which allows for massless particles (like photon)

## Conservation of energy and momentum

We look at conservation laws in collisions

**Elastic collision:** total energy is conserved

**Inelastic collision:** momentum is conserved, not energy

we assume no  
massless  
particles

For a non-relativistic collision

$$\begin{aligned} \sum_i \mathbf{p}_i &= \sum_f \mathbf{p}_f \\ \sum_i \frac{\mathbf{p}_i^2}{2m_i} &= \sum_f \frac{\mathbf{p}_f^2}{2m_f} + Q \\ \sum_i m_i &= \sum_f m_f \end{aligned}$$

also  
conservation of  
charge

where  $i$  and  $f$  refer to initial and final states and  $Q$  to internal energy (for inelastic collisions)

In a relativistic collision we instead have the four equations

$$\sum_i p_i^\mu = \sum_f p_f^\mu$$

This conserves relativistic momentum separately in a given RF, but the conservation of relativistic energy from  $p^0$  is

$$\sum_i \gamma_i m_i c^2 = \sum_f \gamma_f m_f c^2$$

which is a mix of mass and kinetic energy since the relativistic energy can be written as

$$E = \gamma m c^2 = m c^2 + (\gamma - 1) m c^2 = E_0 + K$$

where  $E_0$  is the rest energy and  $K = (\gamma - 1) m c^2$  is the relativistic kinetic energy

In the non-relativistic limit we should keep the first two terms in the expansion of  $\gamma$  in terms of  $v^2/c^2$  giving

$$\sum_i \left( m_i c^2 + \frac{1}{2} m_i v_i^2 \right) = \sum_f m_f c^2 + \frac{1}{2} m_f v_f^2$$

or

$$\sum_i \frac{\mathbf{p}_i^2}{2m_i} = \sum_f \frac{\mathbf{p}_f^2}{2m_f} + \left( \sum_f m_f c^2 - \sum_i m_i c^2 \right)$$

So mass is not conserved, but a combination of kinetic and rest energy is

## Centre-of-mass (CoM)

The CoM in non-relativistic physics is the position vector  $\mathbf{R}$  given by

$$M\mathbf{R} = \sum_i m_i \mathbf{r}_i, \quad M = \sum_i m_i$$

For an isolated (non-accelerated) system we can find a RF where the CoM is at rest so that

$$\mathbf{P} \equiv M\dot{\mathbf{R}} = \sum_i m_i \dot{\mathbf{r}} = 0$$

This is the non-relativistic CoM RF

In relativistic physics we can not define a unique  $\mathbf{R}$ , depends on RF  
However, we can keep CoM RF definition as

$$\mathbf{P} = \sum_i \mathbf{p}_i = 0, \quad \text{where} \quad \mathbf{p}_i = \gamma_i m_i \mathbf{v}_i$$

so the CoM RF is defined by the vector component of the total four-momentum vanishing.

We can always boost to this RF because  $p^\mu$  is time-like, the exception is for a single massless particle where  $p^2 = m^2 c^2 = 0$

Relativistic mechanics breaks elastic/inelastic collision definition because energy and momentum is interchanged by boosts between RFs

**Elastic scattering:** energy is conserved in the CoM RF

common  $\gamma$ -factor *Example:* Two bodies of mass  $m$  collide to form one body of mass  $M$  (fusion) in their centre-of-mass RF ( $\mathbf{p}_1 = -\mathbf{p}_2$ ). Then  $M$  must be at rest and the conservation of energy gives

$$2\gamma mc^2 = Mc^2$$

As a consequence

$$\Delta m = M - 2m = 2m(\gamma - 1) \geq 0$$

so mass increases

Where did the mass come from?

Since  $\Delta mc^2 = 2mc^2(\gamma - 1) = 2K$  it is just the kinetic energy of the initial particles

*Example:* If we have two non-colinear photons (not going in the same direction) with four-momenta

$$p_1^\mu = (p, p, 0, 0) \quad \text{and} \quad p_2^\mu = (p, p \cos \theta, p \sin \theta, 0)$$

the total four-momentum is

$$P^\mu = p_1^\mu + p_2^\mu = (2p, p + p \cos \theta, p \sin \theta, 0)$$

If we square this we get

$$\begin{aligned} P^2 = P_\mu P^\mu &= (2p)^2 - (p + p \cos \theta)^2 - (p \sin \theta)^2 \\ &= 4p^2 - p^2 - 2p^2 \cos \theta - p^2 \cos^2 \theta - p^2 \sin^2 \theta \\ &= 2p^2 - 2p^2 \cos \theta = 2p^2(1 - \cos \theta) > 0 \end{aligned}$$

The “mass”  $M$  for a four-vector  $P^\mu$  that is the sum of multiple four-vectors, given by  $P^2 = M^2 c^2$ , is called the **invariant mass** of the particles