

Example: Doppler effect for light

Assume photon emitted from origin of RF S and detected in RF S' with relative velocity v in the x -direction

The photon is emitted at an angle θ w.r.t. the direction of motion of S' . The photon has momentum $\mathbf{p} = (p \cos \theta, p \sin \theta, 0)$ and energy $E = pc$ giving a four-momentum

$$p^\mu = (p, p \cos \theta, p \sin \theta, 0)$$

in S . In S' it has

$$p'^\mu = (p', p' \cos \theta', p' \sin \theta', 0)$$

By the Lorentz transformation $p'^\mu = L^\mu_\nu p^\nu$ we have

$$p'^0 = \gamma (p^0 - \beta p^1)$$

$$p'^1 = \gamma (p^1 - \beta p^0)$$

$$p'^2 = p^2$$

$$p'^3 = p^3$$

which gives

$$\left. \begin{aligned} p' &= \gamma(p - \beta p \cos \theta) = \gamma p(1 - \beta \cos \theta) \\ p' \cos \theta' &= \gamma(p \cos \theta - \beta p) = \gamma p(\cos \theta - \beta) \\ p' \sin \theta' &= p \sin \theta \\ 0 &= 0 \end{aligned} \right\} \begin{aligned} p' &= \gamma(1 - \beta \cos \theta)p \\ \cos \theta' &= \gamma \frac{p}{p'} (\cos \theta - \beta) \\ &= \gamma \frac{1}{\gamma(1 - \beta \cos \theta)} (\cos \theta - \beta) \\ &= \frac{\cos \theta - \beta}{1 - \beta \cos \theta} \end{aligned}$$

ν is photon
frequency

The de Broglie formula

$$p = \frac{h}{\lambda} = \frac{h\nu}{c}$$

gives

$$\nu' = \gamma(1 - \beta \cos \theta)\nu$$

switch $\beta \rightarrow -\beta$,
 $\theta \leftrightarrow \theta'$, $\nu \leftrightarrow \nu'$

Obviously (?) this can be reversed if the radiation is emitted in S'

$$\nu = \gamma(1 + \beta \cos \theta')\nu'$$

and

$$\cos \theta = \frac{\cos \theta' + \beta}{1 + \beta \cos \theta'}, \quad \sin \theta = \frac{p'}{p} \sin \theta' = \frac{1}{\gamma} \frac{\sin \theta'}{1 + \beta \cos \theta'}$$

In the high boost limit $\beta \simeq 1$ the last equation shows that radiation emitted in S' will have a small angle in S

$$\theta \simeq \frac{1}{\gamma} \frac{\sin \theta'}{1 + \cos \theta'} = \frac{1}{\gamma} \tan \frac{\theta'}{2}$$

Examples: $\theta = \pi$ (receiver moving towards emitter)

$$0 \leq \beta < 1 \quad \nu' = \gamma(1 + \beta)\nu = \frac{1 + \beta}{\sqrt{1 - \beta^2}}\nu = \frac{1 + \beta}{\sqrt{1 - \beta}\sqrt{1 + \beta}} = \sqrt{\frac{1 + \beta}{1 - \beta}}\nu > \nu$$

so higher frequency in S' (**blue-shifted**)
 $\theta = 0$ (receiver moving away from emitter)

$$\nu' = \gamma(1 - \beta)\nu = \sqrt{\frac{1 - \beta}{1 + \beta}}\nu < \nu$$

so lower frequency in S' (**red-shifted**)
 $\theta' = \frac{\pi}{2}$ (received orthogonal to motion)

$$\nu = \gamma\nu' \Rightarrow \nu' = \frac{1}{\gamma}\nu < \nu \quad (\text{red-shifted})$$

S is seen as “slow” from S' (time-dilation)
 $\theta = \frac{\pi}{2}$ (emitter orthogonal to motion)

$$\nu' = \gamma\nu \quad (\text{blue-shifted})$$

here $\cos \theta' = -\beta$, so for $\beta \simeq 1$ we have $\theta' \simeq 180^\circ$

Newton's II (Four-force)

Newton's II law:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

How do we translate to covariant description? (four-vectors)
 Obvious to try

$$K^\mu = \frac{dp^\mu}{d\tau} \quad (\text{four-force})$$

This is a Lorentz vector since p^μ (four-momentum) is
 Can also be related to four-acceleration by

$$K^\mu = \frac{dp^\mu}{d\tau} = m \frac{dU^\mu}{d\tau} = mA^\mu$$

With $K^\mu = (K^0, \mathbf{K})$ and $p^\mu = (E/c, \mathbf{p})$ we have

$$\mathbf{K} = \frac{d\mathbf{p}}{d\tau} = \gamma \frac{d\mathbf{p}}{dt}$$

We define **relativistic force** as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

where $\mathbf{p} = \gamma m \mathbf{v}$ is relativistic momentum, so $\mathbf{K} = \gamma \mathbf{F}$

note the *two* γ factors in $\mathbf{K}$ This has the correct limit when $\gamma \rightarrow 1$ since $\mathbf{p} = \gamma m \mathbf{v} \rightarrow m \mathbf{v}$ then is the non-relativistic momentum

here The real test is of course if it corresponds to reality? Yes!

What is K^0 ?

$$K^0 = \frac{dp^0}{d\tau} = \frac{1}{c} \frac{dE}{d\tau} = \gamma \frac{1}{c} \frac{dE}{dt}$$

From $E^2 = p^2 c^2 + m^2 c^4$ we get

$$2E \frac{dE}{dt} = 2c^2 \mathbf{p} \cdot \frac{d\mathbf{p}}{dt} \Rightarrow E \frac{dE}{dt} = c^2 \mathbf{p} \cdot \frac{d\mathbf{p}}{dt}$$

so

$$\frac{dE}{dt} = c^2 \frac{\mathbf{p}}{E} \cdot \frac{d\mathbf{p}}{dt} = c^2 \frac{\gamma m \mathbf{v}}{\gamma m c^2} \cdot \frac{d\mathbf{p}}{dt} = \mathbf{v} \cdot \frac{d\mathbf{p}}{dt} = \mathbf{v} \cdot \mathbf{F}$$

So K^0 is proportional to the **power** of the (relativistic) force $\mathbf{v} \cdot \mathbf{F}$

$$K^\mu = \left(\frac{\gamma}{c} \mathbf{v} \cdot \mathbf{F}, \gamma \mathbf{F} \right) = \gamma \left(\frac{1}{c} \mathbf{v} \cdot \mathbf{F}, \mathbf{F} \right)$$

Note that $K^\mu U_\mu = m A^\mu U_\mu = 0$ since $A^\mu U_\mu = 0$, so K^μ and U^μ are orthogonal

Since U^μ is time-like K^μ must then be space-like. The RF where K^0 disappears is the instantaneous inertial RF ($\mathbf{v} = 0$) where $K^\mu = (0, \mathbf{F})$.

The Lorentz force

The force on a charged particle in an electromagnetic field

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

For the four-force

$$\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$$

$$K^\mu = \gamma q \left(\frac{1}{c} \mathbf{v} \cdot \mathbf{E}, \mathbf{E} + \mathbf{v} \times \mathbf{B} \right)$$

Would like to write the right-hand side in terms of Lorentz vectors

We use the **electromagnetic field strength tensor** $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

see Problem
Set 6

This can be written in terms of electric and magnetic fields as

$$F^{\mu\nu} = \begin{bmatrix} 0 & -\frac{1}{c}E_1 & -\frac{1}{c}E_2 & -\frac{1}{c}E_3 \\ \frac{1}{c}E_1 & 0 & -B_3 & B_2 \\ \frac{1}{c}E_2 & B_3 & 0 & -B_1 \\ \frac{1}{c}E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

left:four-vector,
right:
three-vector
 $U_\nu = (\gamma c, -\gamma \mathbf{v})$

Here we can notice that $F^{0k} = -\frac{1}{c}E_k$, for $k = 1, 2, 3$, and $F^{kl} = -\epsilon_{klm}B_m$, for $k, l = 1, 2, 3$

We can now write

$$K^0 = \gamma q \frac{\mathbf{v}}{c} \cdot \mathbf{E} = q F^{0\nu} U_\nu$$

notice where we
use three and
four-vector
indices here

and

$$\begin{aligned} K^k &= \gamma q (E_k + \epsilon_{klm} v_l B_m) = \gamma q (c F^{k0} - F^{kl} v_l) \\ &= q (F^{k0} U_0 + F^{kl} U_l) = q F^{k\nu} U_\nu \end{aligned}$$

Thus on four-vector form we can write the Lorentz force

$$K^\mu = q F^{\mu\nu} U_\nu$$

double dot now
means $\frac{d^2}{d\tau^2}$

The resulting equation of motion is, using $K^\mu = m A^\mu = m \frac{d^2 x^\mu}{d\tau^2}$,

$$m \ddot{x}^\mu = q F^{\mu\nu} U_\nu$$

Example: Relativistic charged particle in const. magnetic field

We have

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q\mathbf{v} \times \mathbf{B}$$

where $\mathbf{p} = \gamma m\mathbf{v}$ and $E = \gamma mc^2$ and $\mathbf{B}$ is a constant field

Then

$$\frac{dE}{dt} = \mathbf{F} \cdot \mathbf{v} = q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$$

so the energy is constant (conserved) and $|\mathbf{v}|$ is a constant

Now

$$\frac{d}{dt}(\gamma m\mathbf{v} - q\mathbf{r} \times \mathbf{B}) = 0$$

so we can write

$$\gamma m\mathbf{v} - q\mathbf{r} \times \mathbf{B} = -q\mathbf{r}_0 \times \mathbf{B}$$

where $\mathbf{r}_0$ is some constant vector

The vector $\mathbf{r}_0 \times \mathbf{B}$ is orthogonal to $\mathbf{B}$ so not completely generic, but we can assume no motion in the $\mathbf{B}$ -direction (no force)

We can now write

$$\gamma m\mathbf{v} - q(\mathbf{r} - \mathbf{r}_0) \times \mathbf{B} = 0$$

and a redefinition of the origin gives

$$\gamma m\mathbf{v} - \mathbf{r} \times \mathbf{B} = 0 \quad \text{or} \quad \gamma m\mathbf{v} = \mathbf{r} \times \mathbf{B}$$

Then velocity must be perpendicular to position vector, $\mathbf{r} \cdot \mathbf{v} = 0$, since $\mathbf{r} \cdot (\mathbf{r} \times \mathbf{B}) = 0$, and so

$$\mathbf{r} \cdot \mathbf{v} = \frac{1}{2} \frac{d}{dt} \mathbf{r}^2 = 0$$

and $\mathbf{r}^2$ must also be a constant of motion $\Rightarrow$ we have circular motion

The velocity magnitude is given by

$$v = \frac{q}{\gamma m} |\mathbf{r} \times \mathbf{B}| = \frac{qrB}{\gamma m}$$

The circular frequency is

$$\omega = \frac{v}{r} = \frac{qB}{\gamma m} = \frac{\omega_0}{\gamma}$$

where $\omega_0 = \frac{qB}{m}$ is the non-relativistic cyclotron frequency