

## Covariant Lagrangian and action

$L$  is a scalar, it  
has no  
four-indices

It seems reasonable to write a **Lorentz invariant Lagrangian** for a particle as

$$L(x^\mu(\tau), U^\mu(\tau), \tau)$$

a function of the proper time  $\tau$ , the path  $x^\mu(\tau)$ , and the four-velocity  $U^\mu(\tau) = \frac{dx^\mu}{d\tau}$  of the particle

Here the path  $x^\mu(\tau)$  is the **world-line** of the particle, and  $x^2 > 0$  for all motion (time-like) for a massive particle

The action  $S$  can then be written as the Lorentz invariant functional of the world-line

$$S[x^\mu(\tau)] = \int_{\tau_1}^{\tau_2} L(x^\mu(\tau), U^\mu(\tau), \tau) d\tau$$

where we can find the extremal values,  $\delta S = 0$ , as a function of the path  $x^\mu(\tau)$

The resulting **Lagrange equations** are (by calculus of variations)

four equations!

$$\frac{d}{d\tau} \frac{\partial L}{\partial U^\mu} - \frac{\partial L}{\partial x^\mu} = 0$$

Note:

- 1) The choice of  $\tau$  as independent variable is *not unique*
- 2) We can not use  $\tau$  for a massless particle. Q: Why?  $ds^2 = c^2 d\tau^2 = 0$
- 3) The structure of  $L$  is *not unique*, multiple choices give same e.o.m.
- 4) Can only be used for one particle (only one  $\tau$ )

## Lagrangian for a free particle

**Attempt 1** Assuming the Lagrangian

$$L = \frac{1}{2} m U^\mu U_\mu$$

for a free particle, with  $U^\mu = \frac{dx^\mu}{d\tau}$ , we have

$$\frac{\partial L}{\partial x^\mu} = 0, \quad \frac{\partial L}{\partial U^\mu} = m U_\mu$$

so that by Lagrange's equation

$$\frac{d}{d\tau} (m U_\mu) = \frac{dp_\mu}{d\tau} = 0$$

so no proper  
acceleration  
 $a_0 = 0$

Thus the four-momentum  $p_\mu$  must be a conserved quantity (conservation of relativistic energy and momentum) and the four-acceleration  $A_\mu = \frac{dU_\mu}{d\tau} = 0$

However, without other terms in  $L$ ,  $m$  is a constant that we can freely remove

In fact we can apply the same Lagrangian also for massless particles

$$L = \frac{1}{2} \frac{dx^\mu}{d\lambda} \frac{dx_\mu}{d\lambda}$$

in fact also for  
negative masses

not  $\tau$  if massless

where  $\lambda$  is some (**affine**) parameter that parameterises the motion  $x^\mu(\lambda)$

**Attempt 2** Using  $U^\mu U_\mu = c^2$  we can reformulate the Lagrangian as  $L = amc^2$  where we have inserted a unitless constant  $a$   
Then the action is

$$S = \int_{\tau_1}^{\tau_2} L d\tau = amc^2 \int_{t_1}^{t_2} \frac{d\tau}{dt} dt = amc^2 \int_{t_1}^{t_2} \frac{dt}{\gamma}$$

advantage: can  
be extended to  
multiple  
particles

This is an action with a *non-covariant* Lagrangian  $L = \frac{amc^2}{\gamma}$   
The odd constant can be fixed by looking at

$$L = \frac{amc^2}{\gamma} = amc^2 \left(1 - \frac{1}{2} \frac{v^2}{c^2} + \dots\right) = amc^2 - \frac{1}{2} amv^2 + \dots$$

Here we must have  $a = -1$  to get the correct sign for kinetic energy.  
With this the Lagrangian is

$$L = -\frac{mc^2}{\gamma}$$

The usual Lagrange's equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

gives

$$\frac{\partial L}{\partial x_i} = 0,$$

$$\frac{\partial L}{\partial \dot{x}_i} = -mc^2 \frac{\partial}{\partial \dot{x}_i} \left( \frac{1}{\gamma} \right) = -mc^2 \frac{\partial}{\partial v_i} \sqrt{1 - \frac{v^2}{c^2}} = -mc^2 \frac{1}{2} \left( -2 \frac{v_i}{c^2} \right) \left( 1 - \frac{v^2}{c^2} \right)^{-1/2} = \gamma m v_i$$

so Lagrange's equation gives again conservation of relativistic momentum

$$\frac{d}{dt} (\gamma m v_i) = 0$$

The constant contribution  $-mc^2 = -E_0$  disappears in Lagrange's equation  
The corresponding relativistic Hamiltonian is

$$H = \sum_i \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i - L = \gamma m v^2 + \frac{mc^2}{\gamma} = mc^2 \gamma \left( \frac{v^2}{c^2} + \frac{1}{\gamma^2} \right) = mc^2 \gamma (\beta^2 + 1 - \beta^2) = \gamma mc^2 = E$$

**Attempt 3** –  $\infty$  Since the action

$$S = -mc^2 \int_{\tau_1}^{\tau_2} d\tau$$

in GR called the  
**geodesics**

is proportional to  $cd\tau$  (the length of the world-line/trajectory), finding the path from  $\delta S = 0$ , is the same as finding the minimal or maximal length trajectory of spacetime

An infinite number of non-covariant Lagrangians can now be found using the invariant line element

$$ds^2 = dx^\mu dx_\mu = c^2 d\tau^2$$

compare to: to write

$$\begin{aligned} s &= \int ds = \\ \int \sqrt{dx^2 + dy^2} &= \\ \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt & \end{aligned} \quad S = -mc^2 \int_{\tau_1}^{\tau_2} d\tau = -mc^2 \int_{\tau_1}^{\tau_2} \frac{1}{c} \sqrt{dx^\mu dx_\mu} = -mc \int_{\lambda_1}^{\lambda_2} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$$

for a parameterisation  $\lambda$  of the path  $x^\mu(\lambda)$ , with the Lagrangian

$$L = -mc \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}$$

## Covariant Lagrangian for particle in e.m. field

For a charged particle in an e.m. field the covariant Lagrangian is

$$L = \frac{1}{2} m U^\mu U_\mu + q U_\mu A^\mu$$

where  $A^\mu(x) = (\phi/c, \mathbf{A})$  is the four-potential

We can find Lagrange's equations from

$$\begin{aligned} \frac{\partial L}{\partial x_\mu} &= \partial^\mu (q U_\nu A^\nu) = q U_\nu \partial^\mu A^\nu, \\ \frac{\partial L}{\partial U_\mu} &= m U^\mu + q A^\mu = p^\mu + q A^\mu, \\ \frac{d}{d\tau} \frac{\partial L}{\partial U_\mu} &= \frac{d}{d\tau} (p^\mu + q A^\mu) = K^\mu + q \frac{dA^\mu}{d\tau} = K^\mu + q \frac{dA^\mu}{dx_\nu} \frac{dx_\nu}{d\tau} = K^\mu + q (\partial^\nu A^\mu) U_\nu \end{aligned}$$

giving

$$K^\mu = q U_\nu \partial^\mu A^\nu - q U_\nu \partial^\nu A^\mu = q U_\nu (\partial^\mu A^\nu - \partial^\nu A^\mu) = q U_\nu F^{\mu\nu}$$

the Lagrangian  
is correct  
because it gives  
the correct  
e.o.m.

Which is the same e.o.m. we found in Lecture 17

We call this Lagrangian the **minimal coupling** prescription

We can alternatively start from the non-covariant Lagrangian

$$L = -\frac{mc^2}{\gamma} - U = -\frac{mc^2}{\gamma} - q\phi + q\mathbf{v} \cdot \mathbf{A}$$

arriving at the same answer through

$$\begin{aligned} \frac{\partial L}{\partial x_i} &= -q \frac{\partial \phi}{\partial x_i} + q\mathbf{v} \cdot \frac{\partial \mathbf{A}}{\partial x_i}, \\ \frac{\partial L}{\partial \dot{x}_i} &= \gamma m v_i + q A_i, \\ \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} &= \frac{dp_i}{dt} + q \frac{\partial A_i}{\partial x_j} v_j + q \frac{\partial A_i}{\partial t} \end{aligned}$$

which gives the Lagrange equation

$$\frac{dp_i}{dt} = -q \frac{\partial \phi}{\partial x_i} - q \frac{\partial A_i}{\partial t} + q \left( v_j \frac{\partial A_j}{\partial x_i} - v_j \frac{\partial A_i}{\partial x_j} \right)$$

or

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where we have used

$$\epsilon_{ijk} v_j B_k = \epsilon_{ijk} \epsilon_{klm} v_j \frac{\partial A_m}{\partial x_l} = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) v_j \frac{\partial A_m}{\partial x_l} = v_j \frac{\partial A_j}{\partial x_i} - v_j \frac{\partial A_i}{\partial x_j}$$