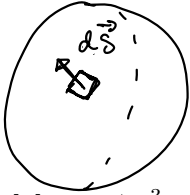


from FYS1120 Short repetition of Maxwell's equations
Electromagnetism

Charge conservation



$$[\rho] = \text{C/m}^3, \\ [\mathbf{j}] = \text{C/m}^2\text{s}$$

Volume V with surface S holds total charge Q and current *out* of volume is I

$$\frac{dQ}{dt} + I = 0 \quad (\text{conservation of charge})$$

With **charge** and **current density**, $\rho(\mathbf{r}, t)$ and $\mathbf{j}(\mathbf{r}, t)$, this is

$$\frac{d}{dt} \int_V \rho(\mathbf{r}, t) dV + \oint_S \mathbf{j}(\mathbf{r}, t) \cdot d\mathbf{S} = 0$$

Using Gauss' theorem we get

$$\int_V \nabla \cdot \mathbf{F} dV = \oint_S \mathbf{F} \cdot d\mathbf{S} \qquad \int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} \right) dV = 0$$

Since this must be true for an arbitrary volume V we must have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$$

which is a **continuity equation** for charge (sometimes called Maxwell's zeroth law)

for multiple
species we have
multiple terms

For a single species of charge we can write its current density in terms of its velocity field:

$$\mathbf{j}(\mathbf{r}, t) = \mathbf{v}(\mathbf{r}, t) \rho(\mathbf{r}, t)$$

For a single charge q with path $\mathbf{r}(t)$ we can write the density in terms of the δ -function

$$\rho(\mathbf{r}, t) = q \delta(\mathbf{r} - \mathbf{r}(t)) \quad [\text{Q: What about units here? A: } \delta(\mathbf{r}) \text{ has units } \text{m}^{-3}]$$

which implies that the current density is

$$\mathbf{j}(\mathbf{r}, t) = q \mathbf{v}(t) \delta(\mathbf{r} - \mathbf{r}(t))$$

where $\mathbf{v}(t)$ is the velocity of the particle

Gauss' law

Given static charge Q , what electric field $\mathbf{E}$ do you get?

Gauss' law (integral form):

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$$

where $\epsilon_0 = 8.85 \cdot 10^{-12} \text{ C}^2/\text{Nm}^2$ is the **permittivity** of vacuum
Written as volume integrals this is

$$\int_V \nabla \cdot \mathbf{E} dV = \int_V \frac{\rho}{\epsilon_0} dV$$

and again, since this should be valid for arbitrary volume we have
Gauss' law (differential form):

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

Example: What is the electric field for a single charge q at rest at $\mathbf{r} = 0$?

Solution for $\mathbf{E}$ must be rotationally symmetric so $\mathbf{E} = E\hat{\mathbf{r}} = \frac{E}{r}\mathbf{r}$

Take S to be a sphere of radius r around charge, then

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = E \oint_S \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dS = 4\pi r^2 E$$

and since

$$\int_V \frac{\rho}{\epsilon_0} dV = \frac{q}{\epsilon_0} \int_V \delta(\mathbf{r}) dV = \frac{q}{\epsilon_0}$$

we have $E = \frac{q}{4\pi\epsilon_0 r^2}$ or

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

which is the **Coulomb field** of a stationary charge

Ampère's law

What magnetic field $\mathbf{B}$ do you get for a current I and a changing electric field $\mathbf{E}$?

Ampère's law (integral form):

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \frac{1}{c^2} \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{S}$$

where C is a closed curve around the current, S is the enclosed surface and $\mu_0 = 4\pi \cdot 10^{-7} \text{ N/A}^2$ is the **permeability** of vacuum (right-hand rule: $d\mathbf{s}$ is counter clockwise)

Using Stoke's theorem this is

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot d\mathbf{s} \quad \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{S} = \int_S \mu_0 \mathbf{j} \cdot d\mathbf{S} + \frac{1}{c^2} \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{S}$$

Again, since this should be true for arbitrary S we have

Ampère's law (differential form):

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j}$$

Note: Changing $\mathbf{E}$ (over time) creates $\mathbf{B}$

$\nabla \cdot (\nabla \times \mathbf{B}) = 0$ It follows that $\nabla \cdot \mathbf{j} = -\frac{1}{c^2 \mu_0} \frac{\partial}{\partial t} \nabla \cdot \mathbf{E}$, and from Gauss' law we then have

$$\nabla \cdot \mathbf{j} + \frac{1}{c^2 \epsilon_0 \mu_0} \frac{\partial \rho}{\partial t} = 0$$

So Ampère + Gauss $\Rightarrow$ charge conservation, and for consistency we must have

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

Gauss' law for a magnetic field

The absence of magnetic monopoles means

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

Gauss' theorem gives

$$\int_V \nabla \cdot \mathbf{B} dV = 0$$

and since the volume is arbitrary we must have

$$\nabla \cdot \mathbf{B} = 0$$

Faraday's law

What electric field $\mathbf{E}$ do you get from a changing magnetic field $\mathbf{B}$?

Faraday's law (integral form):

$$\oint_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$$

where $\mathbf{B}$ is the magnetic field through a closed curve C with surface S

From Stoke's theorem

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S}$$

and with arbitrary surface S we get

Faraday's law (differential form):

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

Note: Changing $\mathbf{B}$ gives $\mathbf{E}$. The two-way coupling gives e.m. waves

Maxwell's equations (in vacuum)

$$\text{i) } \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (\text{source equation})$$

$$\text{ii) } \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j} \quad (\text{source equation})$$

$$\text{iii) } \nabla \cdot \mathbf{B} = 0 \quad (\text{no magnetic source – monopoles})$$

$$\text{iv) } \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (\text{no magnetic source – monopole currents})$$

Magnetic monopole could be included. Equations would then look more symmetrical

Electromagnetic potentials

We can express electric & magnetic fields by a scalar potential $\phi(\mathbf{r}, t)$ and a vector potential $\mathbf{A}(\mathbf{r}, t)$ as

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

Not trivial ! $\mathbf{E}$ & $\mathbf{B}$ have 6 d.o.f., ϕ & $\mathbf{A}$ have 4 d.o.f.

Construction automatically fulfils Maxwell iii) & iv):

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

Rottmann p. 64
 $\nabla \times \nabla\phi = 0$

$$\nabla \times \mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = -\nabla \times \nabla\phi - \nabla \times \frac{\partial\mathbf{A}}{\partial t} + \frac{\partial}{\partial t} \nabla \times \mathbf{A} = 0$$

This means our field theory for e.m. will leave no room for magnetic monopole sources

Maxwell's i) & ii) can be re-written:

i)

$$\nabla \cdot \mathbf{E} = \nabla \cdot \left(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \right) = -\nabla^2\phi - \frac{\partial}{\partial t} \nabla \cdot \mathbf{A}$$

so

$$\nabla^2\phi + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A} = -\frac{\rho}{\epsilon_0}$$

ii)

$$\begin{aligned} \nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial\mathbf{E}}{\partial t} &= \nabla \times (\nabla \times \mathbf{A}) - \frac{1}{c^2} \frac{\partial}{\partial t} \left(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} \right) \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} + \frac{1}{c^2} \frac{\partial}{\partial t} \nabla\phi + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} \end{aligned}$$

where we have used (Rottmann p. 64)

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a}$$

This implies

$$\nabla^2\mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) - \frac{1}{c^2} \frac{\partial}{\partial t} \nabla\phi - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} = -\mu_0\mathbf{j}$$

In total four equations that can be satisfied by the four d.o.f. of ϕ & $\mathbf{A}$