

Gauge choices

The gauge transformations (Lecture 5)

$$\phi \rightarrow \phi' = \phi - \frac{\partial \chi}{\partial t}, \quad \mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla \chi$$

where $\chi = \chi(\mathbf{r}, t)$ is an arbitrary differentiable function, leave $\mathbf{E}$ & $\mathbf{B}$ invariant
The covariant form of the gauge transformations is (Problem Set 6):

$$A^\mu \rightarrow A'^\mu = A^\mu - \partial^\mu \chi$$

where $A^\mu = (\frac{1}{c}\phi, \mathbf{A})$ is the four-potential

Two examples of gauge choices, choose χ so that:

- a) $\nabla \cdot \mathbf{A} = 0$ (**Coulomb gauge**) [typically for non-rel. charged particles]
- b) $\partial_\mu A^\mu = 0$ (**Lorenz gauge**) [typically for relativistic problems]

Coulomb gauge

We can choose χ so that $\nabla^2 \chi = -\nabla \cdot \mathbf{A}$ (a Poisson equation), then the transformed potential fulfils Coulomb gauge

$$\nabla \cdot \mathbf{A}' = \nabla \cdot (\mathbf{A} + \nabla \chi) = \nabla \cdot \mathbf{A} + \nabla^2 \chi = 0$$

It is always possible to find such a χ since the Poisson equation is solvable if $\mathbf{A}$ is well-behaved

In this gauge Maxwell i) & ii) can be written as one equation for ϕ and one equation for $\mathbf{A}$:

$$\text{i) } \nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

No time in equation so non-dynamical, again solvable as a Poisson equation

can be written
 $\square \mathbf{A} = \mu_0 \mathbf{j}_\perp$

$$\text{ii) } \nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{A} = -\mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \phi = -\mu_0 \mathbf{j}_\perp$$

where the **transverse current density** $\mathbf{j}_\perp$ is

$$\mathbf{j}_\perp \equiv \mathbf{j} - \epsilon_0 \frac{\partial}{\partial t} \nabla \phi$$

$\mathbf{j}_\perp$ is called transverse since $\nabla \cdot \mathbf{j}_\perp = 0$

Proof: $\nabla \cdot \mathbf{j}_\perp = \nabla \cdot \mathbf{j} - \epsilon_0 \frac{\partial}{\partial t} \nabla^2 \phi \stackrel{\text{i)}}{=} \nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0$

Maxwell's equations in covariant form

We can use the electromagnetic field strength tensor (see Problem Set 6)

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{bmatrix} 0 & -\frac{1}{c}E_1 & -\frac{1}{c}E_2 & -\frac{1}{c}E_3 \\ \frac{1}{c}E_1 & 0 & -B_3 & B_2 \\ \frac{1}{c}E_2 & B_3 & 0 & -B_1 \\ \frac{1}{c}E_3 & -B_2 & B_1 & 0 \end{bmatrix}$$

to instead write Maxwell i) & ii) as

$$\begin{aligned}
c^2 &= \frac{1}{\mu_0 \epsilon_0} & \text{i) } \nabla \cdot \mathbf{E} &= c \partial_i F^{i0} = \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad \partial_\mu F^{\mu 0} = \frac{\rho}{c \epsilon_0} \quad \Rightarrow \quad \partial_\mu F^{\mu 0} = \mu_0 c \rho \\
F^{ij} &= -\epsilon_{ijk} B_k & \text{ii) } (\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t})_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} B_k - \frac{1}{c^2} \frac{\partial}{\partial t} E_i = -\partial_j F^{ij} - \partial_0 F^{i0} = -\partial_\mu F^{i\mu} = \partial_\mu F^{\mu i} \\
&& &\Rightarrow \partial_\mu F^{\mu i} = \mu_0 \mathbf{j}
\end{aligned}$$

These can be merged to

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu$$

where we have used the **four-current**

$$j^\mu = (c\rho, \mathbf{j})$$

With four-current we can write

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = \partial_0(c\rho) + \partial_i j^i = \partial_\mu j^\mu$$

so conservation of charge can be expressed by

$$\partial_\mu j^\mu = 0$$

What about the Maxwell's iii) and iv)? As we saw above, the underlying construction of the $\mathbf{E}$ and $\mathbf{B}$ fields in terms of the potentials ϕ and $\mathbf{A}$ ensures that they are fulfilled

Lorenz gauge

We can write Maxwell's equations as

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \mu_0 j^\nu$$

and using Lorenz gauge $\partial_\mu A^\mu = 0$ we have

$$\partial_\mu \partial^\mu A^\nu = \mu_0 j^\nu$$

If we further use the d'Alembertian

$$\square = \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2$$

this is what you
should have on
t-shirts

we can write

$$\square A^\nu = \mu_0 j^\nu$$

(Is Lorenz gauge always possible? Yes, the gauge definition is slightly relaxed from Coulomb gauge.)

Dual field

We sometimes need to use the **dual field strength tensor**

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the four-vector version of the Levi-Civita symbol

Under $F^{\mu\nu} \leftrightarrow \tilde{F}^{\mu\nu}$, $\frac{1}{c}\mathbf{E} \rightarrow \mathbf{B}$ and $\mathbf{B} \rightarrow -\frac{1}{c}\mathbf{E}$ so that

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & \frac{1}{c}E_3 & -\frac{1}{c}E_2 \\ B_2 & -\frac{1}{c}E_3 & 0 & \frac{1}{c}E_1 \\ B_3 & \frac{1}{c}E_2 & -\frac{1}{c}E_1 & 0 \end{bmatrix}$$

Proof (Problem Set 6):

$$\begin{aligned} \tilde{F}^{0i} &= \frac{1}{2}\epsilon^{0ijk}F_{jk} = -\frac{1}{2}\epsilon_{ijk}\epsilon_{jkl}B_l = -\frac{1}{2}\epsilon_{jki}\epsilon_{jkl}B_l = -\frac{1}{2}2\delta_{il}B_l = -B_i \\ \tilde{F}^{ij} &= \frac{1}{2}\epsilon^{ij0k}F_{0k} + \frac{1}{2}\epsilon^{ijk0}F_{k0} = -\frac{1}{2}\epsilon_{ijk}\frac{1}{c}E_k - \frac{1}{2}\epsilon_{ijk}\frac{1}{c}E_k = -\frac{1}{c}\epsilon_{ijk}E_k \end{aligned}$$

Then automatically Maxwell's iii) and iv) can be written

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

because the l.h.s. of Maxwell's i) and ii) transform into the l.h.s. of iii) and iv) when switching to the dual tensor