

Lorentz transformation of electromagnetic fields

As a rank-2 tensor the electromagnetic field strength $F^{\mu\nu}$ must transform as

$$F^{\mu\nu} \rightarrow F'^{\mu\nu} = L^\mu_\rho L^\nu_\sigma F^{\rho\sigma}$$

Without loss of generality, let us consider a boost with velocity v in the x -direction

$$L^\mu_\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Then

$$\begin{aligned} F^{\mu\nu} &= -F^{\nu\mu} \\ F^{i0} &= -F^{0i} = \frac{1}{c}E_i \\ F^{ij} &= -\epsilon_{ijk}B_k \end{aligned} \quad \begin{aligned} E'_1 &= cF'^{10} = cL^1_\rho L^0_\sigma F^{\rho\sigma} = cL^1_0 L^0_1 F^{01} + cL^1_1 L^0_0 F^{10} = c(-L^1_0 L^0_1 + L^1_1 L^0_0)F^{10} \\ &= c(-\beta^2\gamma^2 + \gamma^2)\frac{1}{c}E_1 = \gamma^2(1 - \beta^2)E_1 = E_1 \\ E'_2 &= cF'^{20} = cL^2_\rho L^0_\sigma F^{\rho\sigma} = cL^2_2 L^0_0 F^{20} + cL^2_2 L^0_1 F^{21} \\ &= c\gamma\frac{1}{c}E_2 + c(-\beta\gamma)B_3 = \gamma(E_2 - vB_3) \\ E'_3 &= cF'^{30} = cL^3_\rho L^0_\sigma F^{\rho\sigma} = cL^3_3 L^0_0 F^{30} + cL^3_3 L^0_1 F^{31} \\ &= c\gamma\frac{1}{c}E_3 + c(-\beta\gamma)(-B_2) = \gamma(E_3 + vB_2) \end{aligned}$$

We can generalise to fields parallel and transverse to boost $\mathbf{E}_\parallel$ and $\mathbf{E}_\perp$, here

$$\mathbf{E}_\parallel = E_1\mathbf{e}_x, \quad \mathbf{E}_\perp = E_2\mathbf{e}_y + E_3\mathbf{e}_z$$

Then, in general

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel, \quad \mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B})$$

Exercise: Show that

$$\mathbf{B}'_\parallel = \mathbf{B}_\parallel, \quad \mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \frac{\mathbf{v}}{c^2} \times \mathbf{E})$$

Example: In RF S we have $\mathbf{E} = 0$ and $\mathbf{B} = B_0\mathbf{e}_z$. RF S' moves with velocity v in x -direction. What are the fields in S' ?

$$\begin{aligned} \mathbf{E}'_\parallel &= \mathbf{E}_\parallel = 0, \quad \mathbf{B}'_\parallel = \mathbf{B}_\parallel = 0 \\ \mathbf{E}'_\perp &= \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}) = \gamma(0 + v\mathbf{e}_x \times B_0\mathbf{e}_z) = -\gamma v B_0\mathbf{e}_y \\ \mathbf{B}'_\perp &= \gamma(\mathbf{B}_\perp - \frac{\mathbf{v}}{c^2} \times \mathbf{E}) = \gamma\mathbf{B}_\perp = \gamma B_0\mathbf{e}_z \end{aligned}$$

So $\mathbf{E}' = -\gamma v B_0\mathbf{e}_y$ and $\mathbf{B}' = \gamma B_0\mathbf{e}_z$

Example: Current in a neutral wire in different frames

Current I in neutral wire at rest in RF S . Current driven by imposed electric field $\mathbf{E}_0 = E_0 \mathbf{e}_z$ inside wire, assumed uniform, no E outside from charges due to neutrality. We have a circular magnetic field $\mathbf{B}$ given by Ampère's law

$$\oint_C \mathbf{B} \cdot d\mathbf{s} = \mu_0 I$$

This gives at radius r from centre of wire

$$\mathbf{B}(r) = \frac{\mu_0 I}{2\pi r} \mathbf{e}_\phi$$

The current is given from the current density as

$$I = Aj$$

where A is wire area, and $j = v_e \rho_e$ where v_e is the velocity and ρ_e density of electrons in S

We then have ($\parallel$ is in direction of current)

$$E_\parallel = E_0, \quad \mathbf{E}_\perp = 0, \quad B_\parallel = 0, \quad \mathbf{B} = \frac{\mu_0 I}{2\pi r} \mathbf{e}_\phi$$

In RF S' moving with velocity v_e along with electrons (negative z -direction)

$$\begin{aligned} E'_\parallel &= E_\parallel = E_0, & \mathbf{E}'_\perp &= \gamma(\mathbf{E}_\perp + \mathbf{v}_e \times \mathbf{B}) = -\gamma v_e \frac{\mu_0 I}{2\pi r} \mathbf{e}_r \quad (\mathbf{e}_z \times \mathbf{e}_\phi = \mathbf{e}_r) \\ B'_\parallel &= B_\parallel = 0, & \mathbf{B}'_\perp &= \gamma(\mathbf{B}_\perp - \frac{\mathbf{v}_e}{c^2} \times \mathbf{E}) = \gamma \frac{\mu_0 I}{2\pi r} \mathbf{e}_\phi \end{aligned}$$

So stronger B -field, but now E -field out of wire!

What happened to charge neutrality? (no changing B -field to source electric field)

In S , $\rho = 0$ (total) and $j = v_e \rho_e$. Now since $j^\mu = (c\rho, \mathbf{j})$ is a Lorentz vector, in S' we have by a boost in negative z -direction ($v_e < 0$)

$$\begin{aligned} \rho' &= \gamma \left(\rho - \frac{v_e}{c^2} j \right) = -\gamma \frac{v_e^2}{c^2} \rho_e = -\gamma \beta^2 \rho_e \quad (\rho' > 0 \text{ since } \rho_e < 0) \\ j' &= \gamma(j - v_e \rho) = \gamma v_e \rho_e \end{aligned}$$

The latter equation gives a current $I' = \gamma I \Rightarrow \mathbf{B}'_\perp = \gamma \mathbf{B}_\perp$

The charge in a length $\Delta L'$ of wire is $\Delta Q' = \Delta L A \rho'$ in S'

From Gauss' law this gives an electric field E' given by (integrating over wire cylinder)

$$\oint_S \mathbf{E}' \cdot d\mathbf{S}' = \frac{Q'}{\epsilon_0} \Rightarrow E' \cdot 2\pi r \cdot \Delta L' = \frac{\Delta Q'}{\epsilon_0}$$

So

$$E' = \frac{\Delta L' \cdot A \cdot \rho'}{2\pi r \Delta L' \epsilon_0} = -\frac{\gamma \beta^2 \rho_e A}{2\pi r \epsilon_0} = -\gamma \beta \frac{j A}{2\pi r c \epsilon_0} = -\gamma v_e \frac{\mu_0 I}{2\pi r}$$

This is typically a small effect due to small average velocity of electrons

Lorentz invariants

What sort of quantities are Lorentz invariant for e.m. fields?

The simplest thing to try is F^μ_μ but $F^\mu_\mu = 0$ as $F^{\mu\nu}$ is antisymmetric

We can instead calculate

$$\begin{aligned}
 F^{\mu\nu} F_{\mu\nu} &= F^{\mu\nu} g_{\mu\rho} g_{\nu\sigma} F^{\rho\sigma} \\
 &= F^{i0} g_{i\rho} g_{0\sigma} F^{\rho\sigma} + F^{0j} g_{0\rho} g_{j\sigma} F^{\rho\sigma} + F^{ij} g_{i\rho} g_{j\sigma} F^{\rho\sigma} \\
 &= -F^{i0} F^{i0} - F^{0j} F^{0j} + F^{ij} F^{ij} \\
 &= -\frac{1}{c^2} \mathbf{E}^2 - \frac{1}{c^2} \mathbf{E}^2 + 2\mathbf{B}^2 = 2 \left(\mathbf{B}^2 - \frac{1}{c^2} \mathbf{E}^2 \right)
 \end{aligned}$$

where we have used $F^{ij} F^{ij} = (-\epsilon_{ijk} B_k)(-\epsilon_{ijl} B_l) = 2\delta_{kl} B_k B_l$

We can also form

$$\begin{aligned}
 F^{\mu\nu} \tilde{F}_{\mu\nu} &= \frac{1}{2} F^{\mu\nu} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \\
 &= \frac{1}{2} F^{i0} \epsilon_{i0\rho\sigma} F^{\rho\sigma} + \frac{1}{2} F^{0i} \epsilon_{0i\rho\sigma} F^{\rho\sigma} + \frac{1}{2} F^{ij} \epsilon_{ij\rho\sigma} F^{\rho\sigma} \\
 &= \frac{1}{2} F^{i0} \epsilon_{i0jk} F^{jk} - \frac{1}{2} F^{i0} \epsilon_{0ijk} F^{jk} + \frac{1}{2} F^{ij} \epsilon_{ij0k} F^{0k} + \frac{1}{2} F^{ij} \epsilon_{ijk0} F^{k0} \\
 &= -F^{i0} \epsilon_{0ijk} F^{jk} + F^{ij} \epsilon_{ijk0} F^{k0} \\
 &= -F^{i0} \epsilon_{0ijk} F^{jk} - F^{k0} \epsilon_{0kij} F^{ij} \\
 &= -2F^{i0} \epsilon_{0ijk} F^{jk} \\
 &= -4 \left(\frac{1}{c} E_i B_i \right) = -\frac{4}{c} \mathbf{E} \cdot \mathbf{B}
 \end{aligned}$$

where we have used $\epsilon_{0ijk} F^{jk} = -\epsilon_{ijk} F^{jk} = \epsilon_{ijk} \epsilon_{jkl} B_l = 2\delta_{il} B_l = 2B_i$

So

$$\frac{1}{2} F^{\mu\nu} F_{\mu\nu} = \mathbf{B}^2 - \frac{1}{c^2} \mathbf{E}^2 \quad \text{and} \quad \frac{1}{4} F^{\mu\nu} \tilde{F}_{\mu\nu} = -\frac{1}{c} \mathbf{E} \cdot \mathbf{B}$$