

Electromagnetic waves

We consider source free case $j^\mu = 0$, and choose Lorenz gauge $\partial_\mu A^\mu = 0$
 Maxwell's equation(s) reduce to

$$\square A^\mu = 0$$

Q: What sort of equation is this? A wave equation for each component of A^μ

monochromatic:
 one frequency

This has the **plane-wave (monochromatic)** solution

$$A^\mu(x) = A_0^\mu e^{ik_\nu x^\nu}$$

where A_0^μ is a constant four-vector and the **wave four-vector** $k^\mu = (\omega/c, \mathbf{k})$ fulfils

$$k_\nu k^\nu = 0$$

$k_\nu x^\nu = \omega t - \mathbf{k} \cdot \mathbf{r}$
 ω sign can be
 absorbed in $\mathbf{k}$
 direction

Proof: $\partial_\nu \partial^\nu A^\mu = ik^\nu \partial_\nu A^\mu = -k^\nu k_\nu A^\mu = 0$ iff $k_\nu k^\nu = 0$

This implies that $k^\nu k_\nu = \left(\frac{\omega}{c}\right)^2 - \mathbf{k}^2 = 0$ or $\omega = c|\mathbf{k}| = ck$ (dispersion relation)

Here ω is the **(angular) frequency** of the wave and $\mathbf{k}$ is its **wave-number**

Since $\partial_\mu A^\mu = 0$ in Lorenz gauge we also have

$$\partial_\mu A^\mu = -ik_\mu A^\mu = 0 \Rightarrow k_\mu A_0^\mu = 0$$

so k_μ and A_0^μ are orthogonal (in the sense of four-vectors)

Lorenz gauge is achieved by

$$A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \chi \quad \text{where} \quad \partial_\mu \partial^\mu \chi = -\partial_\mu A^\mu$$

since then $\partial_\mu A'^\mu = \partial_\mu A^\mu + \partial_\mu \partial^\mu \chi = 0$

However, $\chi' = \chi + \xi$, where $\partial_\mu \partial^\mu \xi = 0$, also achieves Lorenz gauge since

$$\partial_\mu \partial^\mu \chi' = \partial_\mu \partial^\mu \chi + \partial_\mu \partial^\mu \xi = -\partial_\mu A^\mu$$

only possible
 without sources,
 choose ξ so that
 $\frac{1}{c} \frac{\partial}{\partial t} \xi = -\frac{\phi}{c}$

The remaining (scalar) freedom in ξ can be picked so that $A^0 = 0$, *i.e.* $\phi = 0$

With that choice we are in Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, and Maxwell's equations reduce to $\square \mathbf{A} = 0$ or

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \mathbf{A}(\mathbf{r}, t) = 0$$

The plane-wave solutions can then be written as

again $\omega = ck$

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

where $\nabla \cdot \mathbf{A} = 0$ gives $\mathbf{k} \cdot \mathbf{A}_0 = 0$, meaning $\mathbf{k}$ is transverse to $\mathbf{A}$ and gives the direction of propagation

Note that the formal solutions in both Lorenz and Coulomb gauge are in general complex valued. To find the solutions to a given problem we must add together complex valued solutions to make a real-valued solution

The general solution to a set of initial conditions can be found by a superposition of monochromatic plane-wave-solutions

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \mathbf{A}(\mathbf{k}) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

where the wave-number dependent coefficients $\mathbf{A}(\mathbf{k})$ must fulfil

$$\mathbf{k} \cdot \mathbf{A}(\mathbf{k}) = 0$$

Electric & magnetic fields in e.m. waves

From the potential $A^\mu = (\phi/c, \mathbf{A}) = (0, \mathbf{A}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)})$ we get

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= -\frac{\partial \mathbf{A}}{\partial t} = i\omega \mathbf{A}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} = i\omega \mathbf{A}(\mathbf{r}, t) \\ \mathbf{B}(\mathbf{r}, t) &= \nabla \times \mathbf{A} = i\mathbf{k} \times \mathbf{A}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} = i\mathbf{k} \times \mathbf{A}(\mathbf{r}, t)\end{aligned}$$

Notice that $\mathbf{k} \cdot \mathbf{E} = 0$ since $\mathbf{k} \cdot \mathbf{A} = 0$ and $\mathbf{k} \cdot \mathbf{B} = 0$ since $\mathbf{k} \cdot (\mathbf{k} \times \mathbf{A}) = 0$, so both $\mathbf{E}$ and $\mathbf{B}$ are transverse to the direction of propagation $\mathbf{k}$

Using the unit vector in the direction of propagation $\hat{\mathbf{n}} \equiv \frac{\mathbf{k}}{k}$ we can write

$$\begin{aligned}\mathbf{B} &= i\mathbf{k} \times \mathbf{A} = |\mathbf{k}| i\hat{\mathbf{n}} \times \mathbf{A} = \frac{|\mathbf{k}|}{\omega} \hat{\mathbf{n}} \times \mathbf{E} = \frac{1}{c} \hat{\mathbf{n}} \times \mathbf{E} \\ \mathbf{E} &= -c\hat{\mathbf{n}} \times \mathbf{B}\end{aligned}$$

Rottmann p. 62

where we have used that

$$\begin{aligned}\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \\ \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})\end{aligned}$$

$$c\hat{\mathbf{n}} \times \mathbf{B} = \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{E}) = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{E}) - \mathbf{E}(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) = -\mathbf{E}$$

Also

$$\mathbf{E} \times \mathbf{B} = (-c\hat{\mathbf{n}} \times \mathbf{B}) \times \mathbf{B} = \mathbf{B} \times (c\hat{\mathbf{n}} \times \mathbf{B}) = c\hat{\mathbf{n}}\mathbf{B}^2 - \mathbf{B}(\mathbf{B} \cdot c\hat{\mathbf{n}}) = c\mathbf{B}^2\hat{\mathbf{n}}$$

so $(\mathbf{n}, \mathbf{E}, \mathbf{B})$ forms a right-handed orthogonal set of vectors

Notice that for a monochromatic wave the Lorentz invariants are zero

Rottmann p. 62

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \\ (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - & \\ (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})\end{aligned}$$

$$\begin{aligned}\frac{1}{2}F^{\mu\nu}F_{\mu\nu} &= \mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2 = -(\mathbf{k} \times \mathbf{A})^2 + \frac{1}{c^2}\omega^2\mathbf{A}^2 = -(\mathbf{k} \cdot \mathbf{k})(\mathbf{A} \cdot \mathbf{A}) + (\mathbf{k} \cdot \mathbf{A})^2 + \frac{\omega^2}{c^2}\mathbf{A}^2 = 0 \\ \frac{1}{4}F^{\mu\nu}\tilde{F}_{\mu\nu} &= -\frac{1}{c}\mathbf{E} \cdot \mathbf{B} = \frac{\omega}{c}\mathbf{A} \cdot (\mathbf{k} \times \mathbf{A}) = 0\end{aligned}$$

Polarisation

The monochromatic wave has direction and frequency specified by $\mathbf{k}$. There is still freedom in specifying the direction of $\mathbf{E}$ ($\mathbf{B}$ is then fixed)

This specification is called the **polarisation** of the wave

For the electric field we can write the formal solution

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

where $\mathbf{E}_0 = i\omega \mathbf{A}_0$ is a complex vector, and decompose the real part of $\mathbf{E}$ on two arbitrary (non-parallel) vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ in the plane orthogonal to $\mathbf{k}$ by writing

$$\mathbf{E}_0 = E_{10}e^{i\phi_1}\hat{\mathbf{e}}_1 + E_{20}e^{i\phi_2}\hat{\mathbf{e}}_2$$

below we drop
the

where E_{10} and E_{20} are real amplitudes, giving

Re-qualification

$$\text{Re } \mathbf{E}(\mathbf{r}, t) = E_{10}\hat{\mathbf{e}}_1 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi_1) + E_{20}\hat{\mathbf{e}}_2 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi_2)$$

The vectors $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are called the **polarisation vectors**

Only the relative phase $\phi = \phi_1 - \phi_2$ has physical meaning, a redefinition of t will give a contribution $\Delta\phi$ to both phases so that we can set for example $\phi_1 + \phi_2 = 0$

To investigate the behaviour of the electric field as a function of time we make a redefinition $\phi_2 = \phi'_2 - \frac{\pi}{2}$, using $\cos(x - \frac{\pi}{2}) = \sin x$, so we can write

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= E_{10}\hat{\mathbf{e}}_1 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi_1) + E_{20}\hat{\mathbf{e}}_2 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t + \phi'_2) \\ &= (E_{10} \cos \phi_1 \hat{\mathbf{e}}_1 + E_{20} \sin \phi'_2 \hat{\mathbf{e}}_2) \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) \\ &\quad - (E_{10} \sin \phi_1 \hat{\mathbf{e}}_1 - E_{20} \cos \phi'_2 \hat{\mathbf{e}}_2) \sin(\mathbf{k} \cdot \mathbf{r} - \omega t) \\ &= E'_{10}\hat{\mathbf{e}}'_1 \cos(\mathbf{k} \cdot \mathbf{r} - \omega t) + E'_{20}\hat{\mathbf{e}}'_2 \sin(\mathbf{k} \cdot \mathbf{r} - \omega t)\end{aligned}$$

where we have used the trigonometric identities

$$\begin{aligned}\cos(\theta + \phi_1) &= \cos \theta \cos \phi_1 - \sin \theta \sin \phi_1 \\ \sin(\theta + \phi'_2) &= \sin \theta \cos \phi'_2 + \cos \theta \sin \phi'_2\end{aligned}$$

and defined new amplitudes and polarisation vectors

$$\begin{aligned}E'_{10}\hat{\mathbf{e}}'_1 &= E_{10} \cos \phi_1 \hat{\mathbf{e}}_1 + E_{20} \sin \phi'_2 \hat{\mathbf{e}}_2 \\ E'_{20}\hat{\mathbf{e}}'_2 &= -E_{10} \sin \phi_1 \hat{\mathbf{e}}_1 + E_{20} \cos \phi'_2 \hat{\mathbf{e}}_2\end{aligned}$$

Here $\hat{\mathbf{e}}'_1$ and $\hat{\mathbf{e}}'_2$ are orthogonal if

$$E_{10}^2 \cos \phi_1 \sin \phi_1 = E_{20}^2 \cos \phi'_2 \sin \phi'_2 = E_{20}^2 \cos(\phi_2 + \frac{\pi}{2}) \sin(\phi_2 + \frac{\pi}{2}) = -E_{20}^2 \cos \phi_2 \sin \phi_2$$

using $\cos(\phi_2 + \frac{\pi}{2}) = -\sin \phi_2$ and $\sin(\phi_2 + \frac{\pi}{2}) = \cos \phi_2$

except when
 $\phi_1 = 0$ and
 $\phi_2 = 0$

We can now exploit that we can always redefine $t = 0$ to shift ϕ_1 and ϕ_2 so that this is fulfilled

Over time the electric field $\mathbf{E}(\mathbf{r}, t)$ now rotates forming an ellipse with axis $\hat{\mathbf{e}}'_1$ and $\hat{\mathbf{e}}'_2$, and amplitude E'_{10} and E'_{20} , in the plane perpendicular to the propagation, giving rise to what we call **elliptic polarisation**

Some special cases can be mentioned

- **Circular polarisation:** If the amplitudes are equal, $E'_{10} = E'_{20}$, the electric field has a circular motion in the transverse plane.
- **Linear polarisation:** If $\phi_1 = \phi_2 = 0$ (or $\phi_1 = -\phi_2 = \pi$) the ellipse collapses to a line giving

$$\begin{aligned}E'_{10}\hat{\mathbf{e}}'_1 &= E_{10}\hat{\mathbf{e}}_1 + E_{20} \sin \frac{\pi}{2} \hat{\mathbf{e}}_2 = E_{10}\hat{\mathbf{e}}_1 + E_{20}\hat{\mathbf{e}}_2 \\ E'_{20}\hat{\mathbf{e}}'_2 &= E_{20} \cos \frac{\pi}{2} \hat{\mathbf{e}}_2 = 0\end{aligned}$$

Thus the electric field oscillates on a fixed axis orthogonal to $\mathbf{k}$ given by $E_{10}\hat{\mathbf{e}}_1 + E_{20}\hat{\mathbf{e}}_2$. In this case $\mathbf{k}$ and $\mathbf{E}$ span what we call the **polarisation plane**