

Stationary sources

We now look at e.m. with **static sources**, meaning time-independent charge and current density

$$j^\mu(\mathbf{r}) = (\rho(\mathbf{r}), \mathbf{j}(\mathbf{r}))$$

This fixes an inertial RF which we call the **laboratory frame**

Maxwell's equations $\square A^\mu = \mu_0 j^\mu$ then have time-independent solutions $A^\mu(\mathbf{r})$, and Lorenz gauge, $\partial_\mu A^\mu = 0$, reduces to Coulomb gauge $\nabla \cdot \mathbf{A} = 0$. Maxwell's equations are then (Lecture 20):

the transverse current is equal to the current	i)	$\underbrace{\nabla^2 \phi = -\frac{\rho}{\epsilon_0}}_{\text{electrostatics}}$	ii)	$\underbrace{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}}_{\text{magnetostatics}}$
		gives time-independent electric field		gives time-independent magnetic field
		$\mathbf{E}(\mathbf{r}) = -\nabla \phi(\mathbf{r})$		$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r})$

These are both equations on the Poisson form $\nabla^2 f = g$

Electrostatics

Since the Poisson equations are linear differential equations, if we can write $\rho = \sum_i \rho_i$ then the solution is $\phi = \sum_i \phi_i$, where $\nabla^2 \phi_i = -\frac{\rho_i}{\epsilon_0}$

For a point charge q at $\mathbf{r} = 0$ we have $\rho = q\delta(\mathbf{r})$. The resulting Poisson equation

$$\nabla^2 \phi = -\frac{q}{\epsilon_0} \delta(\mathbf{r})$$

is tricky to solve directly, however, from Gauss' law (Lecture 19) we know that

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} \quad \Rightarrow \quad \phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r}$$

$$\nabla = \hat{\mathbf{r}} \frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}} \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} + \hat{\boldsymbol{\theta}} \frac{1}{r} \frac{\partial}{\partial \theta}$$

since with this ϕ we get

$$\mathbf{E} = -\nabla \phi = -\hat{\mathbf{r}} \frac{\partial}{\partial r} \phi = \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

$1/4\pi r$ is the
**Green's
function** for
 ∇^2

This means that

$$\nabla^2 \left(\frac{1}{4\pi r} \right) = -\delta(\mathbf{r})$$

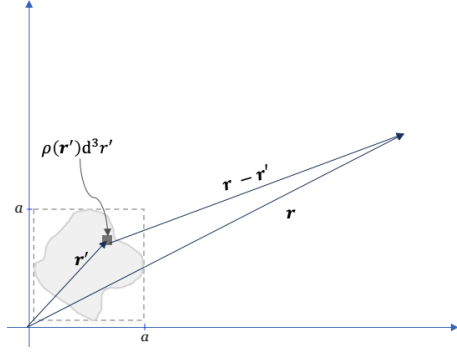
We can then show that for a general charge distribution $\rho(\mathbf{r})$

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r'$$

$$\begin{aligned} \nabla^2 \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} &= \\ \nabla^2 \frac{1}{4\pi r''} &= \\ -\delta(\mathbf{r}'') & \end{aligned}$$

is the solution since

$$\nabla^2 \phi(\mathbf{r}) = \frac{1}{\epsilon_0} \int \rho(\mathbf{r}') \nabla^2 \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} d^3 r' = -\frac{1}{\epsilon_0} \int \rho(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') d^3 r' = -\frac{\rho(\mathbf{r})}{\epsilon_0}$$



The corresponding electric field is

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi = -\frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} d^3r' = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}') d^3r'$$

The above solutions assume infinite volume and $\phi(\mathbf{r}) \rightarrow 0$ as $r \rightarrow \infty$

For a finite volume we need boundary conditions on ϕ (Dirichlet) or $\mathbf{E}$ (von Neuman), for example equipotential at the surface of a conductor

Then the solution is $\phi(\mathbf{r}) = \phi_c(\mathbf{r}) + \phi_0(\mathbf{r})$ where $\nabla^2\phi_0 = 0$ is fixed so that the boundary conditions are satisfied

Multipole expansion

How do we (practically) find ϕ and $\mathbf{E}$ when we are far from charge distribution?

We can change variables to the small (unitless) position $\boldsymbol{\xi} = \frac{\mathbf{r}'}{r}$

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = (r^2 - 2\mathbf{r} \cdot \mathbf{r}' + r'^2)^{1/2} = \frac{1}{r} \left(1 - 2\frac{\mathbf{r}}{r} \cdot \boldsymbol{\xi} + \xi^2\right)^{-1/2} \equiv \frac{1}{r} f(\boldsymbol{\xi})$$

and do a Taylor expansion of f around $\boldsymbol{\xi} = 0$, giving

$$\frac{\partial f}{\partial \xi_i} = -\frac{1}{2} f^3 \left(-2\frac{r_i}{r} + 2\xi_i\right) \quad f(\xi) = f(0) + \sum_i \xi_i \frac{\partial f}{\partial \xi_i} \Big|_{\xi=0} + \frac{1}{2} \sum_{ij} \xi_i \xi_j \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} \Big|_{\xi=0} + \dots$$

$$\frac{\partial^2 f}{\partial \xi_i \partial \xi_j} = -\frac{3}{2} f^2 f' \left(-2\frac{r_i}{r} + 2\xi_i\right) - \delta_{ij} f^3 = 1 + \sum_i \xi_i \frac{r_i}{r} + \frac{1}{2} \sum_{ij} \xi_i \xi_j \left(3\frac{r_i}{r} \frac{r_j}{r} - \delta_{ij}\right) + \dots$$

This gives

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \left[1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{1}{2} \left(3\frac{(\mathbf{r} \cdot \mathbf{r}')^2}{r^4} - \left(\frac{r'}{r}\right)^2\right) + \dots\right]$$

and for the potential we can then write the general solution as

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0 r} \int \rho(\mathbf{r}') \left[1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{1}{2} \left(3\frac{(\mathbf{r} \cdot \mathbf{r}')^2}{r^4} - \left(\frac{r'}{r}\right)^2\right) + \dots\right] d^3r' \\ &= \phi_0(\mathbf{r}) + \phi_1(\mathbf{r}) + \phi_2(\mathbf{r}) + \dots \end{aligned}$$

Monopole term:

$$\phi_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r} \int \rho(\mathbf{r}') d^3 r' = \frac{Q}{4\pi\epsilon_0 r}$$

choose origin
carefully

Contribution as if the total charge Q was at $r = 0$

Dipole term:

$$\phi_1(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3} \int \rho(\mathbf{r}') (\mathbf{r} \cdot \mathbf{r}') d^3 r' = \frac{\hat{\mathbf{n}} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2}$$

where we have introduced the direction to the observer $\hat{\mathbf{n}} \equiv \frac{\mathbf{r}}{r}$, and the **dipole moment**

$$\mathbf{p} \equiv \int \rho(\mathbf{r}) \mathbf{r} d^3 r$$

Quadrupole term:

$$\phi_2(\mathbf{r}) = \frac{1}{8\pi\epsilon_0 r^3} \int \rho(\mathbf{r}') (3(\hat{\mathbf{n}} \cdot \mathbf{r}')^2 - r'^2) d^3 r' \equiv \frac{Q_{\hat{\mathbf{n}}}}{8\pi\epsilon_0 r^3}$$

where $Q_{\hat{\mathbf{n}}}$ is the **quadrupole moment** about $\hat{\mathbf{n}}$

The electric field can be expanded in the same way

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 + \mathbf{E}_2 + \dots \quad \text{with} \quad \mathbf{E}_n = -\nabla \phi_n$$

Monopole term:

$$\mathbf{E}_0 = -\nabla \phi_0 = \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}}$$

which is the Coulomb field of a charge Q at the origin

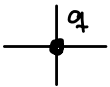
Dipole term:

Exercise: Shows
last equality

$$\mathbf{E}_1 = -\nabla \phi_1 = -\nabla \left(\frac{\mathbf{r} \cdot \mathbf{p}}{4\pi\epsilon_0 r^3} \right) = \frac{1}{4\pi\epsilon_0 r^3} (3(\hat{\mathbf{n}} \cdot \mathbf{p})\hat{\mathbf{n}} - \mathbf{p})$$

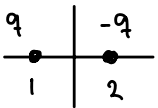
Example:

only monopole $Q = q$ since



$$\mathbf{p} = \int \rho(\mathbf{r}) \mathbf{r} d^3 r = q \int \delta(\mathbf{r}) \mathbf{r} d^3 r = 0$$

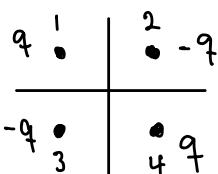
no monopole since $Q = q - q = 0$



$$\mathbf{p} = \int [q\delta(\mathbf{r} - \mathbf{r}_1)\mathbf{r} - q\delta(\mathbf{r} - \mathbf{r}_2)\mathbf{r}] d^3 r = q(\mathbf{r}_1 - \mathbf{r}_2)$$

no monopole since $Q = q - q + q - q = 0$

no dipole since



$$\mathbf{p} = q(\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{r}_3 + \mathbf{r}_4) = 0$$

but does have a quadrupole

Magnetostatics

same diff. eq. as
electrostatics

The equation of magnetostatics, $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}(\mathbf{r})$, has the solutions

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3 r'$$

The corresponding magnetic field is

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \left(\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \times \mathbf{j}(\mathbf{r}') d^3 r'$$

Here, with the change of variables $\mathbf{r}'' = \mathbf{r} - \mathbf{r}'$,

$$\nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \nabla'' \frac{1}{r''} = \hat{\mathbf{r}}'' \frac{\partial}{\partial r''} \frac{1}{r''} = -\frac{\hat{\mathbf{r}}''}{r''^2} = -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

thus

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3 r'$$