

Electromagnetic momentum density

Momentum conservation between source and field tells us that

$$\frac{d}{dt}(\mathbf{p}_f + \mathbf{p}_s) = 0$$

Since for a single charge q

$$\frac{d\mathbf{p}_s}{dt} = \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

we have

$$\begin{aligned} \frac{d}{dt}\mathbf{p}_f &= -\frac{d}{dt}\mathbf{p}_s = -\int (\rho\mathbf{E} + \mathbf{j} \times \mathbf{B})dV \\ &= -\int \left[\epsilon_0(\nabla \cdot \mathbf{E})\mathbf{E} + \frac{1}{\mu_0} \left(\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right) \times \mathbf{B} \right] dV \\ &= -\int \left[\epsilon_0\mathbf{E} \times (\nabla \times \mathbf{E}) - \frac{1}{\mu_0}(\nabla \cdot \mathbf{B})\mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} \right] dV \\ &= \int \left[\epsilon_0\mathbf{E} \times \frac{\partial \mathbf{B}}{\partial t} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} \right] dV \\ &= \frac{d}{dt} \int \epsilon_0 \mathbf{E} \times \mathbf{B} dV \end{aligned}$$

Gauss' law
 $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$
 and Ampère's
 law
 Faraday's law
 and no magnetic
 monopoles

proof in
 Appendix C.2 in
 Leinaas' book

Here we have used for both $\mathbf{E}$ and $\mathbf{B}$ that

$$\mathbf{E} \times (\nabla \times \mathbf{E}) = (\nabla \cdot \mathbf{E})\mathbf{E} - \sum \hat{\mathbf{e}}_i \cdot \nabla (E_i \mathbf{E} - \frac{1}{2} E^2 \hat{\mathbf{e}}_i)$$

where the total derivative term disappears at the surface of the volume
 This means that the field **momentum density** is

$$\mathbf{g} = \epsilon_0 \mathbf{E} \times \mathbf{B} \quad \left[\text{or} \quad \mathbf{g} = \frac{1}{c^2} \mathbf{S} \right]$$

Q: What are the units of $\mathbf{g}$? Momentum/ m^3

Energy-momentum tensor

Since

$$F^{0i} = -\frac{1}{c} E_i \quad \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2} \left(\mathbf{B}^2 - \frac{1}{c^2} \mathbf{E}^2 \right) \quad \text{and} \quad F^{0\nu} F^0_{\nu} = F^{0\nu} g_{\nu\rho} F^{0\rho} = -\frac{1}{c^2} \mathbf{E}^2$$

we can write

$$u = \frac{1}{\mu_0} \frac{1}{2} \left(\mathbf{B}^2 + \frac{1}{c^2} \mathbf{E}^2 \right) = \frac{1}{\mu_0} \left(-F^{0\nu} F^0_{\nu} + \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right)$$

This is a component of the **energy-momentum tensor** for an e.m. field

$$T^{\mu\nu} \equiv \frac{1}{\mu_0} \left(-F^{\mu\rho} F^{\nu}_{\rho} + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right)$$

where $T^{00} = u$

Further, this tensor contains Poynting's vector:

$$F^{ij} = -\epsilon_{ijk} B_k \quad T^{0i} = \frac{1}{\mu_0} (-F^{0\rho} F^i_{\rho}) = \frac{1}{\mu_0} (-F^{0j} F^i_j) = -\frac{1}{\mu_0} \left(\frac{1}{c} E_j \epsilon_{ijk} B_k \right) = -\frac{1}{c} S_i$$

The remaining entries T^{ij} are called the **Maxwell stress tensors**

Example: Einstein's field equations (general relativity) are:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

Example: Energy and momentum density of a plane wave

We consider a plane wave where

$$\mathbf{B} = \frac{1}{c} \hat{\mathbf{n}} \times \mathbf{E} \quad \text{and} \quad \mathbf{E} = -c \hat{\mathbf{n}} \times \mathbf{B}$$

where $\hat{\mathbf{n}} = \frac{\mathbf{k}}{|\mathbf{k}|}$ is a unit vector in the propagation direction

Here

$$\begin{aligned} u &= \frac{1}{2} \left(\epsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0 c^2} (\hat{\mathbf{n}} \times \mathbf{E})^2 \right) \\ &= \frac{1}{2} (\epsilon_0 \mathbf{E}^2 + \epsilon_0 (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) (\mathbf{E} \cdot \mathbf{E}) - \underbrace{\epsilon_0 (\hat{\mathbf{n}} \cdot \mathbf{E})^2}_{=0}) \\ &= \frac{1}{2} (\epsilon_0 \mathbf{E}^2 + \epsilon_0 \mathbf{E}^2) = \epsilon_0 \mathbf{E}^2 \end{aligned}$$

Rottmann p. 62

with equal contribution to the energy (density) from electric & magnetic fields
Poynting's vector is

Rottmann p. 62

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \frac{1}{\mu_0 c} \mathbf{E} \times (\hat{\mathbf{n}} \times \mathbf{E}) = \epsilon_0 c [\hat{\mathbf{n}} (\mathbf{E} \cdot \mathbf{E}) - \mathbf{E} (\mathbf{E} \cdot \hat{\mathbf{n}})] \\ &= \epsilon_0 c \hat{\mathbf{n}} \mathbf{E}^2 = u c \hat{\mathbf{n}} \quad [\text{compare to } \mathbf{j} = \rho \mathbf{v}] \end{aligned}$$

So Poynting's vector is pointed in the direction of propagation and transports energy at speed c

Example: Energy of a static point charge

Since there is no **B**-field the energy density is $u = \frac{1}{2} \epsilon_0 E^2$ where the field is the Coulomb field ($\mathbf{r}_s$ is charge position)

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}_s}{|\mathbf{r} - \mathbf{r}_s|^3}$$

which gives

$$u(\mathbf{r}, t) = \frac{q^2}{32\pi^2\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}_s|^4} \rightarrow \infty \quad \text{as} \quad \mathbf{r} \rightarrow \mathbf{r}_s$$

The total energy has the same problem

$$\mathbf{r}' = \mathbf{r} - \mathbf{r}_s$$

$$E_{se} = \int u d^3r = \frac{q^2}{32\pi^2\epsilon_0} 4\pi \int_0^\infty \frac{1}{r'^4} r'^2 dr' = \frac{q^2}{8\pi\epsilon_0} \left[-\frac{1}{r'} \right]_0^\infty = \infty$$

This is called the **self-energy term** and is a problem for all point charges

first introduced
by Lorentz and
Abraham in the
early 1900s

The problem can be solved by **regularisation**, e.g. cut-off of integral at $r' = a$ so that

$$E_a = \frac{q^2}{8\pi\epsilon_0 a}$$

How do we deal with the extra energy? We perform what is called **renormalisation** by splitting the energy

$$E_{se} = E_a + E(r < a)$$

not the only
option, we can
for example also
redefine q

and incorporating the divergent energy into the rest mass

$$m = m_b + m_{se}$$

as $m_{se} = \frac{E_{se}}{c^2}$, where $m_b = -\frac{E(r < a)}{c^2}$ is the **bare** (imagined) mass without the field
For example for an electron the cut-off radius r_e which makes the resulting mass equal to the electron mass m_e is given as

$$m_e = \frac{E_a}{c^2} = \frac{e^2}{8\pi\epsilon_0 r_e c^2} \Rightarrow r_e = \frac{e^2}{8\pi\epsilon_0 m_e c^2}$$

which is the **classical electron radius**

Preamble

In general, if L is a linear differential operator, $G(x, y)$ in

$$LG(x, y) = \delta(x - y)$$

is its **Green's function**

Working on the electrostatic equation for a single charge we found that

some
disagreement in
literature where
to keep $-$ and
 4π

$$\nabla^2 \left(\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \right) = -\delta(\mathbf{r} - \mathbf{r}')$$

Here

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|}$$

is the Green's function of ∇^2

This allowed us to build the general solution for a charge density $\rho(\mathbf{r})$

Time-dependent sources solution

We want to solve Maxwell's equations with a time dependent source

$$\square A^\nu(\mathbf{r}, t) = \mu_0 j^\nu(\mathbf{r}, t)$$

so we need to solve four equations of the form

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) f(\mathbf{r}, t) = s(\mathbf{r}, t) \quad (*)$$

for f given s . To proceed we introduce the Fourier transforms $\tilde{f}$ and $\tilde{s}$

$$f(\mathbf{r}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(\mathbf{r}, \omega) e^{-i\omega t} d\omega, \quad \tilde{f}(\mathbf{r}, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\mathbf{r}, t) e^{i\omega t} dt$$

Inserted in (*) this gives

$$\left(\nabla^2 + \frac{\omega^2}{c^2} \right) \tilde{f} = -\tilde{s} \quad (\text{Helmholtz' equation})$$

If $\tilde{s} = \delta(\mathbf{r})$ then $\tilde{f}(\mathbf{r}, \omega)$ is the Green's function of $\nabla^2 + \frac{\omega^2}{c^2}$

We can show that this Green's function is

$$\tilde{f}(\mathbf{r}, \omega) = \frac{e^{\pm i\omega r/c}}{4\pi r}$$

Proof: We start by isolating the $1/r$ singularity writing $\alpha = \pm \frac{\omega}{c}$

$$\left(\nabla^2 + \frac{\omega^2}{c^2} \right) \frac{e^{\pm i\omega r/c}}{4\pi r} = (\nabla^2 + \alpha^2) \frac{e^{i\alpha r}}{4\pi r} = (\nabla^2 + \alpha^2) \left(\frac{1}{4\pi r} - \frac{1 - e^{i\alpha r}}{4\pi r} \right)$$

Rottmann p. 69:

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right) + \dots$$

For the non-singular term we get

$$\begin{aligned} \nabla^2 \left(\frac{1 - e^{i\alpha r}}{r} \right) &= \frac{1}{r^2} \frac{\partial}{\partial r} \left[r^2 \frac{\partial}{\partial r} \left(\frac{1 - e^{i\alpha r}}{r} \right) \right] = \frac{1}{r^2} \frac{\partial}{\partial r} [-1 - i\alpha r e^{i\alpha r} + e^{i\alpha r}] \\ &= \frac{1}{r^2} [-i\alpha e^{i\alpha r} + \alpha^2 r e^{i\alpha r} + i\alpha e^{i\alpha r}] = \alpha^2 \frac{e^{i\alpha r}}{r} \end{aligned}$$

So we have

$$(\nabla^2 + \alpha^2) \frac{e^{i\alpha r}}{4\pi r} = -\delta(\mathbf{r}) - \alpha^2 \frac{e^{i\alpha r}}{4\pi r} + \alpha^2 \frac{e^{i\alpha r}}{4\pi r} = -\delta(\mathbf{r})$$