

We can now use the Green's function to write a general solution to Helmholtz' equation for any $\tilde{s}$ as

$$\tilde{f}_{\pm}(\mathbf{r}, \omega) = \frac{1}{4\pi} \int \frac{e^{\pm i\omega|\mathbf{r}-\mathbf{r}'|/c}}{|\mathbf{r}-\mathbf{r}'|} \tilde{s}(\mathbf{r}', \omega) d^3r'$$

This is the correct solution since

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{f}_{\pm}(\mathbf{r}, \omega) = \frac{1}{4\pi} \int -4\pi\delta(\mathbf{r}-\mathbf{r}') \tilde{s}(\mathbf{r}', \omega) d^3r' = -\tilde{s}(\mathbf{r}, \omega)$$

Rewinding the Fourier transformation gives the solution of the original equation

$$\begin{aligned} f_{\pm}(\mathbf{r}, t) &= \frac{1}{4\pi} \int \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega(t \mp |\mathbf{r}-\mathbf{r}'|/c)} \tilde{s}(\mathbf{r}', \omega) d\omega \right) \frac{1}{|\mathbf{r}-\mathbf{r}'|} d^3r' \\ &= \frac{1}{4\pi} \int \frac{s(\mathbf{r}', t \mp |\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} d^3r' \end{aligned}$$

The solution looks like the solution for static charges, but the source has a different time

$$t_{\pm} = t \pm |\mathbf{r}-\mathbf{r}'|/c$$

than the observer t

The solution f_- is called the **retarded solution** since $t_- < t$ and the effect of the source is delayed – it is the source from some time ago

The solution f_+ is called the **advanced solution**

We usually use the retarded solution working from a set of initial conditions, but advanced solution can be used when working from final conditions

Retarded potential

Since the specific equations we want to solve are

$$c^2 = \frac{1}{\epsilon_0 \mu_0} \quad \square\left(\frac{\phi}{c}, \mathbf{A}\right) = \mu_0(c\rho, \mathbf{j}) \quad \Rightarrow \quad \square\phi = \frac{\rho}{\epsilon_0}, \quad \text{and} \quad \square\mathbf{A} = \mu_0\mathbf{j}$$

the resulting scalar & vector potentials are

$$\begin{aligned} \phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_-)}{|\mathbf{r}-\mathbf{r}'|} d^3r' \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t_-)}{|\mathbf{r}-\mathbf{r}'|} d^3r' \end{aligned}$$

These integrals have contributions only from the past light-cone of the point $(ct, \mathbf{r})$

We can see this since $(ct, \mathbf{r})$ and $(ct_-, \mathbf{r}')$ have a light-like distance

$$\Delta s^2 = (ct - ct_-)^2 - (\mathbf{r} - \mathbf{r}')^2 = (ct - ct_- - c|\mathbf{r}-\mathbf{r}'|/c)^2 - (\mathbf{r} - \mathbf{r}')^2 = 0$$

Potential of a point charge

observer $\mathbf{r}$,
charge $\mathbf{r}(t)$ The change and current density of a point charge q moving on path $\mathbf{r}(t)$ is

$$\begin{aligned}\rho(\mathbf{r}, t) &= q\delta(\mathbf{r} - \mathbf{r}(t)) \\ \mathbf{j}(\mathbf{r}, t) &= q\mathbf{v}(t)\delta(\mathbf{r} - \mathbf{r}(t))\end{aligned}$$

The scalar potential is then

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta(\mathbf{r}' - \mathbf{r}(t_-))}{|\mathbf{r} - \mathbf{r}'|} d^3r'$$

where $t_- = t - |\mathbf{r} - \mathbf{r}'|/c$

four coordinates in play: $\mathbf{r}, \mathbf{r}(t), \mathbf{r}', \mathbf{r}''$ To perform the integral we change coordinates to $\mathbf{r}'' = \mathbf{r}' - \mathbf{r}(t_-)$ with

$$d^3r'' = |J|d^3r'$$

where J is the Jacobian matrix of the transformation given by

$$\begin{aligned}J_{kl} &= \frac{\partial x''_k}{\partial x'_l} = \delta_{kl} - \frac{dx_k}{dt_-} \frac{\partial t_-}{\partial x'_l} = \delta_{kl} - v_k(t_-) \left(-\frac{1}{c} \frac{\partial}{\partial x'_l} |\mathbf{r} - \mathbf{r}'| \right) \\ &= \delta_{kl} + \frac{1}{c} v_k(t_-) \frac{\partial}{\partial x'_l} \sqrt{r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}'} \\ &= \delta_{kl} - \frac{1}{c} v_k(t_-) \frac{x_l - x'_l}{|\mathbf{r} - \mathbf{r}'|}\end{aligned}$$

With $\boldsymbol{\beta}(t_-) = \frac{\mathbf{v}(t_-)}{c}$ and $\hat{\mathbf{n}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$ we can write $J_{kl} = \delta_{kl} - \beta_k \hat{n}_l$
Some (tedious) work shows that

$$|J| = 1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}$$

The scalar potential is then

$$\mathbf{r}'' = 0 \Rightarrow \mathbf{r}' = \mathbf{r}(t_-) \quad \phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \int \frac{\delta(\mathbf{r}'')}{|\mathbf{r} - \mathbf{r}'|} \frac{1}{1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}}} d^3r'' = \frac{q}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}(t_-)| (1 - \boldsymbol{\beta}(t_-) \cdot \hat{\mathbf{n}}(t_-))}$$

or

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 (R - \boldsymbol{\beta} \cdot \mathbf{R})_{\text{ret}}}$$

where $\mathbf{R}(t) = \mathbf{r} - \mathbf{r}(t)$ and “ret” indicates that the time is t_-

The vector potential is similarly found to be

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 qc}{4\pi} \left(\frac{\boldsymbol{\beta}}{R - \boldsymbol{\beta} \cdot \mathbf{R}} \right)_{\text{ret}}$$

These are the **Lienard-Wiechert potentials**

If we choose as RF the rest frame of the charge at t_- , then $\boldsymbol{\beta}(t_-) = 0$ so

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 R_{\text{ret}}} \quad \text{and} \quad \mathbf{A}(\mathbf{r}, t) = 0$$

mening we have a Coulomb potential with distance $R_{\text{ret}} = |\mathbf{r} - \mathbf{r}(t)|$

Potentials far away from sources

We now look at the potentials far away ($\mathbf{r}$) from the charges ($\mathbf{r}'$), $r \gg r'$

Using the series expansion of

$$\sqrt{1+x} = 1 + \frac{1}{2}x + \dots$$

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{r^2 + r'^2 - 2\mathbf{r} \cdot \mathbf{r}'} = r \sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\frac{\mathbf{r} \cdot \mathbf{r}'}{r^2}} = r \left(1 + \frac{1}{2} \left(\frac{r'}{r}\right)^2 - \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \dots\right)$$

keeping only terms of $\mathcal{O}(\frac{r'}{r})$ and below we can then approximate the retarded time as

$$t_- = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c} = t - \frac{r}{c} + \frac{\mathbf{r} \cdot \mathbf{r}'}{cr} + \mathcal{O}\left(\left(\frac{r'}{r}\right)^2\right) \simeq t_r + \frac{\mathbf{r} \cdot \mathbf{r}'}{rc}$$

where $t_r \equiv t - \frac{r}{c}$ is the **retarded time of the origin** of the source

The scalar potential expanded around t_r is

$$\rho(\mathbf{r}', t_-) = \rho(\mathbf{r}', t_r) + (t_- - t_r) \left. \frac{\partial \rho(\mathbf{r}, t_-)}{\partial t_-} \right|_{t_- = t_r} + \dots \simeq \rho(\mathbf{r}', t_r) + \frac{\mathbf{r} \cdot \mathbf{r}'}{rc} \left. \frac{\partial \rho(\mathbf{r}, t_-)}{\partial t_-} \right|_{t_- = t_r}$$

where we have dropped both higher order $\frac{\partial}{\partial t_-}$ and $\frac{r'}{r}$ terms

Since we found in the multipole expansion (Lecture 23)

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \left[1 + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2} + \mathcal{O}\left(\left(\frac{r'}{r}\right)^2\right) \right]$$

the integrand in the solution for the potential can be written

$$\frac{\rho(\mathbf{r}', t_-)}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{r} \rho(\mathbf{r}', t_r) + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^2 c} \frac{\partial \rho(\mathbf{r}', t_r)}{\partial t_r} + \dots$$

where in total we have kept terms up to $1/r$ except terms with higher time-derivatives

This gives

$$\begin{aligned} \phi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}', t_-)}{|\mathbf{r} - \mathbf{r}'|} d^3r' \\ &= \frac{1}{4\pi\epsilon_0 r} \int \rho(\mathbf{r}', t_r) d^3r' + \frac{1}{4\pi\epsilon_0 r^2 c} \int (\mathbf{r} \cdot \mathbf{r}') \frac{\partial \rho(\mathbf{r}', t_r)}{\partial t_r} d^3r' + \dots \\ &= \frac{Q}{4\pi\epsilon_0 r} + \frac{\mathbf{r} \cdot \dot{\mathbf{p}}_{\text{ret}}}{4\pi\epsilon_0 r^2 c} + \dots \end{aligned}$$

compare to
static result
 $\frac{Q}{4\pi\epsilon_0 r} + \frac{\mathbf{r} \cdot \mathbf{p}}{4\pi\epsilon_0 r^3}$

where $\dot{\mathbf{p}}_{\text{ret}}$ is the time derivative of the dipole moment at t_r

We can do the same expansion for the vector potential

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}', t_-)}{|\mathbf{r} - \mathbf{r}'|} d^3r' \\ &= \frac{\mu_0}{4\pi r} \underbrace{\int \mathbf{j}(\mathbf{r}', t_r) d^3r'}_{\dot{\mathbf{p}}_{\text{ret}} \text{ [Lecture 24]}} + \frac{\mu_0}{4\pi r^2 c} \int (\mathbf{r} \cdot \mathbf{r}') \frac{\partial \mathbf{j}(\mathbf{r}', t_r)}{\partial t_r} d^3r' + \dots \end{aligned}$$

The second integral can be worked out from a result in Lecture 24

$$\int (\mathbf{r} \cdot \mathbf{r}') \mathbf{j}(\mathbf{r}', t_r) d^3 r' = \mathbf{m} \times \mathbf{r} + \frac{1}{2} \frac{d}{dt} \int \mathbf{r}' (\mathbf{r} \cdot \mathbf{r}') \rho(\mathbf{r}', t_r) d^3 r'$$

Defining

$$\mathbf{D}_{\hat{n}} \equiv \int \mathbf{r}' (\mathbf{r}' \cdot \hat{\mathbf{n}}) \rho(\mathbf{r}', t) d^3 r'$$

where we use $\hat{\mathbf{n}} \equiv \mathbf{r}/r$, we have

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r} \left(\dot{\mathbf{p}} + \frac{1}{c} \dot{\mathbf{m}} \times \hat{\mathbf{n}} + \frac{1}{2c} \ddot{\mathbf{D}}_{\hat{n}} + \dots \right)_{\text{ret}}$$