

Radiation fields

the parenthesis
does not depend
on r explicitly
but through t_r

$$\begin{aligned}\nabla &= \hat{n} \frac{\partial}{\partial r} + \dots \\ &\text{gives} \\ \nabla \frac{1}{r} &= -\frac{1}{r^2} \hat{n} \text{ and} \\ \nabla t_r &= \\ \nabla(t - \frac{r}{c}) &= -\frac{1}{c} \hat{n}\end{aligned}$$

The resulting magnetic field is

$$\begin{aligned}\mathbf{B}(\mathbf{r}, t) &= \nabla \times \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \nabla \frac{1}{r} \times \left(\dot{\mathbf{p}} + \frac{1}{c} \dot{\mathbf{m}} \times \hat{n} + \frac{1}{2c} \ddot{\mathbf{D}}_{\hat{n}} + \dots \right)_{\text{ret}} \\ &\quad + \frac{\mu_0}{4\pi r} \nabla t_r \times \frac{d}{dt_r} \left(\dot{\mathbf{p}} + \frac{1}{c} \dot{\mathbf{m}} \times \hat{n} + \frac{1}{2c} \ddot{\mathbf{D}}_{\hat{n}} + \dots \right)_{\text{ret}} \\ &= \frac{\mu_0}{4\pi r^2} \left(\dot{\mathbf{p}} + \frac{1}{c} \dot{\mathbf{m}} \times \hat{n} + \frac{1}{2c} \ddot{\mathbf{D}}_{\hat{n}} + \dots \right)_{\text{ret}} \times \hat{n} \\ &\quad + \frac{\mu_0}{4\pi r c} \left(\ddot{\mathbf{p}} + \frac{1}{c} \ddot{\mathbf{m}} \times \hat{n} + \frac{1}{2c} \ddot{\mathbf{D}}_{\hat{n}} \right) \times \hat{n}\end{aligned}$$

A similar equation can be found for $\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}$

For large r it is the second term that dominates giving rise to **radiation fields**, where we can argue that far from the sources we have plane waves so that

$$\begin{aligned}\mathbf{B}_{\text{rad}}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi r c} \left(\ddot{\mathbf{p}} + \frac{1}{c} \ddot{\mathbf{m}} \times \hat{n} + \frac{1}{2c} \ddot{\mathbf{D}}_{\hat{n}} \right)_{\text{ret}} \times \hat{n} \\ \mathbf{E}_{\text{rad}}(\mathbf{r}, t) &= -c\hat{n} \times \mathbf{B}_{\text{rad}} = \frac{\mu_0}{4\pi r} \left((\ddot{\mathbf{p}} \times \hat{n}) \times \hat{n} - \frac{1}{c} \ddot{\mathbf{m}} \times \hat{n} + \frac{1}{2c} (\ddot{\mathbf{D}}_{\hat{n}} \times \hat{n}) \times \hat{n} \right)_{\text{ret}}\end{aligned}$$

Here we have used that $\ddot{\mathbf{m}}$ becomes a very confused vector:

$$((\ddot{\mathbf{m}} \times \hat{n}) \times \hat{n}) \times \hat{n} = -\ddot{\mathbf{m}} \times \hat{n}$$

since from Rottmann

$$\begin{aligned}\hat{n} \times (\mathbf{m} \times \hat{n}) &= \mathbf{m}(\hat{n} \cdot \hat{n}) - \hat{n}(\mathbf{m} \cdot \hat{n}) \\ (\mathbf{m} \times \hat{n}) \times \hat{n} &= -\mathbf{m} + \hat{n}(\mathbf{m} \cdot \hat{n})\end{aligned}$$

Radiation field scales

Derivation of radiation fields depends on assumptions on scales:

- $r \gg r' \sim a$ where a is “size” of sources
- $r \gg \lambda$ where λ is wavelength of radiation

The latter is important because fields contain (two) time derivative of momenta so are proportional to (angular) frequency ω and the factor $1/rc$, which if $\lambda > r$ gives

$$\frac{\omega}{rc} = \frac{2\pi}{\lambda r} < \frac{1}{r^2}$$

so that the $1/r^2$ terms with one time derivative that we threw away become important

If also $\lambda \gg a$ then $\ddot{p}, \ddot{m} \gg \ddot{D}$ since the extra time-derivative means it is proportional to $\omega \sim \frac{c}{\lambda}$ and the D -term goes as a^2 compared to a for the dipole terms

If we compare electric & magnetic dipoles:

$$\ddot{\mathbf{p}} \propto \rho \propto q, \quad \ddot{\mathbf{m}} \propto j \propto qv \Rightarrow \frac{1}{c}\ddot{\mathbf{m}} \propto q\frac{v}{c} \ll q \text{ if } v \ll c$$

So for non-relativistic sources electric dipole radiation tends to dominate over magnetic dipoles

The $\ddot{\mathbf{p}}$ term is often referred to as E1, the $\ddot{\mathbf{m}}$ term M1 and $\ddot{\mathbf{D}}$ as E2

Electric dipole radiation (E1)

When E1 dominates we have

$$\mathbf{E}_{\text{rad}}(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0 r c^2} (\ddot{\mathbf{p}}_{\text{ret}} \times \hat{\mathbf{n}}) \times \hat{\mathbf{n}}, \quad \mathbf{B}_{\text{rad}}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r c} \ddot{\mathbf{p}}_{\text{rad}} \times \hat{\mathbf{n}}(*)$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

Poynting's vector for the electric dipole is

$$\begin{aligned} \mathbf{S}(\mathbf{r}, t) &= \frac{1}{\mu_0} \mathbf{E}_{\text{rad}} \times \mathbf{B}_{\text{rad}} = -\frac{c}{\mu_0} (\hat{\mathbf{n}} \times \mathbf{B}_{\text{rad}}) \times \mathbf{B}_{\text{rad}} = \frac{c}{\mu_0} \hat{\mathbf{n}} B_{\text{rad}}^2 - \frac{c}{\mu_0} \mathbf{B}_{\text{rad}} (\underbrace{\hat{\mathbf{n}} \cdot \mathbf{B}_{\text{rad}}}_{=0 \text{ free field}}) \\ &= \frac{c}{\mu_0} \frac{\mu_0^2}{16\pi^2 r^2 c^2} (\ddot{\mathbf{p}}_{\text{ret}} \times \hat{\mathbf{n}})^2 \hat{\mathbf{n}} = \frac{\mu_0}{16\pi^2 r^2 c} (\ddot{p}^2 \sin^2 \theta)_{\text{ret}} \hat{\mathbf{n}} \end{aligned}$$

where θ is the angle between $\ddot{\mathbf{p}}$ and $\hat{\mathbf{n}}$ (direction to observer)

Since $\mathbf{S}$ gives the energy current density we see radiation is directed in $\hat{\mathbf{n}}$ -direction away from source

If we look at the power P emitted through an infinitesimal area dA of a sphere Σ surrounding the source this is

$$\begin{aligned} [P] &= \text{J/s}, \\ [S] &= \text{J/m}^2\text{s} \end{aligned}$$

$$dP = \mathbf{S} \cdot d\mathbf{A} = \frac{\mu_0}{16\pi^2 c r^2} \ddot{p}_{\text{ret}}^2 \sin^2 \theta \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} dA = \frac{\mu_0}{16\pi^2 c r^2} \ddot{p}_{\text{ret}}^2 \sin^2 \theta r^2 d\Omega = \frac{\mu_0}{16\pi^2 c} \ddot{p}_{\text{ret}}^2 \sin^2 \theta d\Omega$$

where $d\Omega = \sin \theta d\theta d\phi$ is the solid angle of the area

We can write this as a differential power per unit solid angle

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c} \ddot{p}_{\text{ret}}^2 \sin^2 \theta$$

use variable substitution

$$u = \cos \theta$$

Integrating over the sphere gives the total power

$$P(t) = \int \frac{dP}{d\Omega} d\Omega = \frac{\mu_0}{16\pi^2 c} \ddot{p}_{\text{ret}}^2 \int_0^{2\pi} \int_0^\pi \sin^2 \theta \sin \theta d\theta d\phi = \frac{\mu_0}{8\pi c} \ddot{p}_{\text{ret}}^2 \int_0^\pi \sin^3 \theta d\theta = \frac{\mu_0}{6\pi c} \ddot{p}_{\text{ret}}^2$$

Larmor's formula

A point charge with path $\mathbf{r}(t)$ has the electric dipole moment

$$\mathbf{p}(t) = \int \mathbf{r} \rho(\mathbf{r}, t) d^3 r = q \int \mathbf{r} \delta(\mathbf{r} - \mathbf{r}(t)) d^3 r = q\mathbf{r}(t)$$

so $\ddot{\mathbf{p}}(t) = q\ddot{\mathbf{r}}(t) = q\mathbf{a}(t)$

If the charge is non-relativistic then contribution from M1 is small so radiation fields can be approximated as E1

Far away from the source we then have the radiated power

$$P(t) = \frac{\mu_0}{6\pi c} \dot{p}_{\text{ret}}^2 = \frac{\mu_0 q^2}{6\pi c} a_{\text{ret}}^2$$

This is called **Larmor's formula**

Power can be shown to be a Lorentz invariant so in the relativistic case the acceleration can be substituted by the proper acceleration a_0 giving

$$P(t) = \frac{\mu_0 q^2}{6\pi c} a_0^2 = \frac{\mu_0 q^2}{6\pi c} \left(\gamma^4 a^2 + \gamma^6 \frac{(\mathbf{v} \cdot \mathbf{a})^2}{c^2} \right)$$

where $\mathbf{v}$ and $\mathbf{a}$ is the velocity and acceleration of the particle in some RF S

What is the magnetic dipole moment $\mathbf{m}$ here anyways?

$$\begin{aligned} \mathbf{m} &= \frac{1}{2} \int \mathbf{r} \times \mathbf{j} d^3r = \frac{1}{2} \int \mathbf{r} \times q\mathbf{v}(t) \delta(\mathbf{r} - \mathbf{r}(t)) d^3r \\ &= \frac{q}{2} \mathbf{r}(t) \times \mathbf{v}(t) = \frac{q}{2m} \mathbf{r}(t) \times m\mathbf{v}(t) = \frac{q}{2m} \mathbf{L}(t) \end{aligned}$$

In QM we often write $\boldsymbol{\mu} = \frac{q}{2m} \mathbf{L}$

Example: Scattering on a free charged particle

We will investigate **Thomson scattering**, the scattering of a free e.m. plane-wave on a free electron of charge e and mass m_e

Assume wave propagates in z -direction with linear polarisation in x -direction, then

$$\mathbf{E}(\mathbf{r}, t) = E_0 \hat{\mathbf{e}}_x \cos \omega t, \quad \mathbf{B}(\mathbf{r}, t) = \frac{1}{c} \hat{\mathbf{e}}_z \times \mathbf{E} = \frac{E_0}{c} \hat{\mathbf{e}}_y \cos \omega t,$$

Poynting's vector (the energy current) for the incoming wave is

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} = \frac{1}{\mu_0} E_0 \hat{\mathbf{e}}_x \cos \omega t \times \frac{E_0}{c} \hat{\mathbf{e}}_y \cos \omega t = c \epsilon_0 E_0^2 \cos^2 \omega t \hat{\mathbf{e}}_z$$

The electron feels a force from the electric field

$$\mathbf{F}_e = e\mathbf{E} = eE_0 \hat{\mathbf{e}}_x \cos \omega t$$

From the magnetic field $F_m = |e\mathbf{v} \times \mathbf{B}| = eE_0 |\cos \omega t| \frac{v}{c}$, which we ignore under the assumption $v \ll c$

With initial conditions $v = 0$ at $t = 0$ the motion in the x -direction is then

$$v \ll c \text{ if } \frac{eE_0}{m_e \omega} \ll 1$$

$$a_x = \frac{F_e}{m_e} = \frac{eE_0}{m_e} \cos \omega t \quad \Rightarrow \quad v_x = \frac{eE_0}{m_e \omega} \sin \omega t \quad \Rightarrow \quad x = -\frac{eE_0}{m_e \omega^2} \cos \omega t$$

The power radiated *from* the electron per unit solid angle is then

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c} \dot{p}_{\text{ret}}^2 \sin^2 \Theta = \frac{\mu_0 e^2}{16\pi^2 c} a_{\text{ret}}^2 \sin^2 \Theta = \frac{\mu_0 e^4}{16\pi^2 m_e^2 c} E_0^2 \cos^2 \omega t_r \sin^2 \Theta$$

where Θ is angle between direction of dipole acceleration (x -direction) and observer
We can relate this angle to the spherical coordinates of the observer $\mathbf{r} = (r, \theta, \phi)$ by

$$\sin^2 \Theta = 1 - \sin^2 \theta \cos^2 \phi$$

Using the **classical electron radius** $r_0 = e^2/(4\pi\epsilon_0 m_e c^2) = 2.82 \cdot 10^{-15} \text{ m}$, we get

$$\begin{aligned} \frac{dP(\theta, \phi, t)}{d\Omega} &= \frac{e^4}{16\pi^2 \epsilon_0^2 m_e^2 c^4} c\epsilon_0 E_0^2 \cos^2 \omega t_r (1 - \sin^2 \theta \cos^2 \phi) \\ &= r_0^2 c\epsilon_0 E_0^2 \cos^2 \omega t_r (1 - \sin^2 \theta \cos^2 \phi) \end{aligned}$$

We can now integrate over the solid angle to get the total radiated power, however, we already know from Larmor's formula that

$$P(t) = \frac{\mu_0 q^2}{6\pi c} a_{\text{ret}}^2 = \frac{\mu_0 e^4}{6\pi m_e^2 c} E_0^2 \cos^2 \omega t_r = \frac{8\pi}{3} c\epsilon_0 r_0^2 E_0^2 \cos^2 \omega t_r$$

can also write
 $\sigma = \frac{8\pi}{3} \left(\frac{\lambda_c}{2\pi}\right)^2 \alpha^2$
where α is the
fine structure
constant and λ_c
is the Compton
wavelength

The amount of energy (per unit time) scattered by the particle, divided by the incoming flux of energy (energy per unit time per unit area) is used to define the **(Thomson scattering) cross section** σ as

$$\sigma \equiv \frac{P}{S} = \frac{\frac{8\pi}{3} c\epsilon_0 r_0^2 E_0^2 \cos^2 \omega t_r}{c\epsilon_0 E_0^2 \cos^2 \omega t_r} = \frac{8\pi}{3} r_0^2$$

Notice how this has units of area. This is interpreted as an *effective* area for the electron measuring its ability to “eat” radiation

1 barn = For an electron this is numerically $\sigma = 6.65 \cdot 10^{-29} \text{ m}^2$ or 0.665 barn

10^{-28} m^2 This can easily be generalised to a particle with charge $q = Ze$ and mass m as

$$\sigma = \frac{8\pi}{3} Z^4 \left(\frac{m_e}{m}\right)^2 r_0^2$$

The cross section can also be generalised to other fields, and to QM and QFT where the cross section is a ratio of probabilities (probability current)

We can similarly find the differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{S} \frac{dP}{d\Omega} = \frac{c\epsilon_0 r_0^2 E_0^2 \cos^2 \omega t_r (1 - \sin^2 \theta \cos^2 \phi)}{c\epsilon_0 E_0^2 \cos^2 \omega t_r} = r_0^2 (1 - \sin^2 \theta \cos^2 \phi)$$

We can average over the polarisation direction ϕ (xy -plane) to get the result for unpolarised incoming e.m. wave:

Rottmann
p. 143

$$\int_0^{2\pi} (1 - \sin^2 \theta \cos^2 \phi) d\phi = 2\pi - \pi \sin^2 \theta = 2\pi(1 - \frac{1}{2} \sin^2 \theta) = 2\pi \cdot \frac{1}{2} (1 + \cos^2 \theta)$$

Thus

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_0^2 (1 + \cos^2 \theta)$$