

Reminder of Lagrangian mechanics

Lagrangian $L(q, \dot{q}, t)$ of a system is given in terms of **generalised coordinates** q and **velocities** $\dot{q}$, with possible explicit time dependence t , as difference between kinetic and potential energy

$$L = K - V$$

The **action integral** (functional) of the path $q(t)$ is

$$S[q(t)] = \int_{t_i}^{t_f} L(q, \dot{q}, t) dt \quad (1)$$

The **equations of motion** are found as the stationary points of the action, $\delta S = 0$, by varying the path

$$q(t) \rightarrow q'(t) = q(t) + \delta q(t), \quad (2)$$

with fixed $\delta q(t_i) = \delta q(t_f) = 0$

This gives the **Lagrange equations**

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, 2, \dots, d, \quad (3)$$

where $q = \{q_1, q_2, \dots, q_d\}$ is the set of coordinates and d the number of d.o.f.

Translating to fields

we generalise
from scalar ϕ to
other fields later

How do we deal with a *classical* field $\phi(x) = \phi(\mathbf{r}, t)$ with no specific position, but defined everywhere?

Divide space into cells ΔV_i with centre coordinate $\mathbf{r}_i$ and let generalised coordinates be $q_i = \phi(\mathbf{r}_i, t)$

The dynamical quantity is now the *field value*

The generalised velocities are then

$$\dot{q}_i = \frac{d}{dt} \phi(\mathbf{r}_i, t) = \dot{\phi}(\mathbf{r}_i, t)$$

The total Lagrangian can be written as the sum of contributions from each cell
 $L_i = K_i - V_i$

$$L = \sum_i L_i(q_i, \dot{q}_i, q_{i'}, t) = \sum_i \mathcal{L}_i \Delta V_i, \quad (4)$$

where $\mathcal{L}_i = L_i / \Delta V_i$ is the **Lagrangian density** in cell i

including higher
derivatives is
called a higher
derivative field
theory (doh!)

The L_i can depend on the field in the nearest neighbouring cells $q_{i'}$ because energy may depend on field gradient $\nabla \phi$

We can now take continuum limit $\Delta V_i \rightarrow d^3 r$

$$L = \int \mathcal{L}(\phi, \dot{\phi}, \nabla \phi, \mathbf{r}, t) d^3 r, \quad (5)$$

$$[\mathcal{L}] = \text{J/m}^3$$

where $\mathcal{L}$ is the **Lagrangian density** at a point $\mathbf{r}$ and time t

The dependence on $\mathbf{r}$ is to pick up explicit dependence on cell index i

1/c can be
ignored

Inserted into the action integral (1) we get

$$S[\phi(x)] = \int_{t_i}^{t_f} \int \mathcal{L}(\phi, \dot{\phi}, \nabla\phi, \mathbf{r}, t) d^3\mathbf{r} dt = \frac{1}{c} \int_{\Omega} \mathcal{L}(\phi(x), \partial_{\mu}\phi(x), x) d^4x, \quad (6)$$

it can be shown
that d^4x is
Lorentz
invariant

where the integral is over a space-time volume Ω and $d^4x = c dt dx dy dz$
The e.o.m. for ϕ can now be found as the stationary points of this action
Note:

see relativistic
mechanics
discussion

- This is easily extended to other fields component by component. For example a vector field $A_{\nu}(x)$ has the Lagrangian density $\mathcal{L}(A_{\nu}(x), \partial_{\mu}A_{\nu}(x), x_{\mu})$
- We require the Lagrangian density $\mathcal{L}$ to be Lorentz invariant. This will lead to the equations of motion for ϕ being Lorentz covariant
- The Hamiltonian is *not* Lorentz invariant in general, which is a reason to prefer working with the Lagrangian in a relativistic field theory.

Equations of Motion

We find e.o.m. for field by Hamilton's principle:

find the field $\phi(x)$ where the action is a stationary point, $\delta S = 0$

Vary field as

$$\phi(x) \rightarrow \phi'(x) = \phi(x) + \delta\phi(x) \quad (7)$$

where we assume variation $\delta\phi(x) = 0$ on boundary $\partial\Omega$ of volume Ω

This gives a change in the action for fixed x

$$\begin{aligned} \delta S &= S' - S = \int_{\Omega} (\mathcal{L}' - \mathcal{L}) d^4x = \int_{\Omega} \delta\mathcal{L} d^4x \\ &= \int_{\Omega} \left(\frac{\partial\mathcal{L}}{\partial\phi} \delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta(\partial_{\mu}\phi) \right) d^4x \end{aligned} \quad (8)$$

We can write

$$\delta(\partial_{\mu}\phi) = \partial_{\mu}\phi' - \partial_{\mu}\phi = \partial_{\mu}(\phi' - \phi) = \partial_{\mu}\delta\phi \quad (9)$$

and add and subtract the same term to the integrand

$\delta\phi$ is function of
 x only

$$\begin{aligned} \delta S &= \int_{\Omega} \left(\frac{\partial\mathcal{L}}{\partial\phi} \delta\phi - \frac{d}{dx^{\mu}} \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta\phi + \frac{d}{dx^{\mu}} \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \partial_{\mu}\delta\phi \right) d^4x \\ &= \int_{\Omega} \left(\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dx^{\mu}} \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \right) \delta\phi d^4x + \int_{\Omega} \frac{d}{dx^{\mu}} \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta\phi \right) d^4x \end{aligned} \quad (10)$$

We evaluate the second integral by using the fundamental theorem of calculus

$$\int_{\Omega} \frac{d}{dx^{\mu}} \left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta\phi \right) d^4x \sim \left[\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)} \delta\phi \right]_{\partial\Omega} = 0 \quad (11)$$

where we get zero because $\delta\phi$ is zero on the surface

To get $\delta S = 0$, since the choice of $\delta\phi(x)$ is arbitrary, we must have

$$\frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad (12)$$

which is the **Lagrange equation of motion for fields**.

This equation generalises to fields with indices, giving one equation per index value, *e.g.* for a vector field $A_\nu(x)$ the equations are

$$\frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0, \quad \nu = 0, 1, 2, 3. \quad (13)$$