

Generalised Momentum and Hamiltonian Density

in mechanics

$$p \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

notice extra

factor of c in

this definition

We define **generalised momentum density** $\pi(x)$ of field $\phi(x)$ by

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}$$

The Legendre transformation to the **Hamiltonian density** $\mathcal{H}$ gives

$$\mathcal{H}(\phi, \nabla \phi, \pi, x) \equiv \pi(x) \partial_0 \phi(x) - \mathcal{L}(\phi, \partial_\mu \phi, x), \quad \text{with} \quad H \equiv \int \mathcal{H} d^3r$$

We assume that $\pi(x)$ is solved for $\partial_0 \phi$ and used to remove it from Hamiltonian

The generalised momentum and Hamiltonian are *not* Lorentz covariant/invariant

Generalised momentum can be written on covariant form in the **De Donder-Weyl formalism** by defining

$$\pi^\mu(x) \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

and a corresponding Lorentz invariant Hamiltonian density can be constructed

Noether's Theorem for Classical Fields

We consider transformations of coordinates x^μ and field $\phi(x)$ that leave the Lagrange equation form invariant

$$\begin{aligned} x^\mu &\rightarrow x'^\mu \equiv x^\mu + \delta x^\mu(\phi, x), \\ \phi(x) &\rightarrow \phi'(x) \equiv \phi(x) + \delta \phi(\phi, x). \end{aligned}$$

Criterion of form invariance similar to mechanics

F has same role
as total
time-derivative
function in
mechanics

$$\mathcal{L}[\phi', \partial'_\mu \phi', x'] \left| \frac{\partial(ct', x', y', z')}{\partial(ct, x, y, z)} \right| = \mathcal{L}[\phi, \partial_\mu \phi, x] + \frac{d}{dx^\mu} F^\mu$$

where $F^\mu(\phi, x)$

Assuming transformations are infinitesimal Jacobian evaluates to

$$\left| \frac{\partial(ct', x', y', z')}{\partial(ct, x, y, z)} \right| = \det(\delta_\nu^\mu + \partial_\nu \delta x^\mu) = 1 + \partial_\mu \delta x^\mu$$

and we expand left-hand Lagrangian to first order in infinitesimals as

$$\mathcal{L}[\phi, \partial_\mu \phi, x] + \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) + \partial_\mu \mathcal{L} \delta x^\mu$$

Must be careful with $\delta(\partial_\mu \phi)$ since we are varying both field and coordinates

$$\delta(\partial_\mu \phi) \equiv \partial'_\mu \phi' - \partial_\mu \phi = \partial_\mu \phi' - (\partial_\mu \delta x^\nu) \partial_\nu \phi' - \partial_\mu \phi = \partial_\mu \delta \phi - \partial_\nu \phi (\partial_\mu \delta x^\nu),$$

where we use that to first order in infinitesimals

$$\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = \frac{\partial}{\partial x^\mu} - \frac{\partial \delta x^\nu}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = \partial_\mu - (\partial_\mu \delta x^\nu) \partial_\nu$$

Keeping terms to first order in infinitesimals we have

$$\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu \delta \phi - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi \partial_\mu \delta x^\nu + \mathcal{L} \partial_\mu \delta x^\mu + \partial_\mu \mathcal{L} \delta x^\mu = \frac{d}{dx^\mu} \delta F^\mu$$

where $\delta F(\delta \phi, \delta x^\mu)$ is F written in terms of infinitesimal to first order

symmetry
before e.o.m.

$\delta x^\mu(\phi, x)$ and $\delta \phi(\phi, x)$ must fulfil this to be an **off-shell symmetry**
Using

$$\begin{aligned} \frac{d\mathcal{L}}{dx^\mu} &= \frac{\partial \mathcal{L}}{\partial \phi} \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \partial_\mu \partial^\nu \phi + \partial_\mu \mathcal{L} \\ &= \left(\frac{d}{dx^\nu} \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \right) \partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \partial_\mu \partial^\nu \phi + \partial_\mu \mathcal{L} \\ &= \frac{d}{dx^\nu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\nu \phi)} \partial_\mu \phi \right) + \partial_\mu \mathcal{L}, \end{aligned}$$

to remove $\partial_\mu \mathcal{L}$ and the Lagrange equation, the **on-shell symmetry** is

$$\begin{aligned} &\left(\frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu \delta \phi - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi \partial_\mu \delta x^\nu + \mathcal{L} \partial_\mu \delta x^\mu + \frac{d\mathcal{L}}{dx^\mu} \delta x^\mu \\ &- \frac{d}{dx^\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi \right) \delta x^\nu = \frac{d}{dx^\mu} \delta F^\mu \end{aligned}$$

We pull out the total derivative and get

$$\frac{d}{dx^\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi - \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\nu \phi \delta x^\nu + \mathcal{L} \delta x^\mu - \delta F^\mu \right) = 0$$

change to
partial assuming
e.o.m. solved for
 $\phi(x)$ and
substituted into
current so it is
explicitly a
function of x
only

or using generalised momentum

$$\partial_\mu (\pi^\mu \delta \phi - \pi^\mu \partial_\nu \phi \delta x^\nu + \mathcal{L} \delta x^\mu - \delta F^\mu) = 0$$

This is **Noether's theorem for fields**

Defining the **Noether four-current**

$$\delta N^\mu \equiv \pi^\mu (\delta \phi - \partial_\nu \phi \delta x^\nu) + \mathcal{L} \delta x^\mu - \delta F^\mu$$

Noether's theorem is a continuity equation

$$\frac{1}{c} \frac{\partial N^0}{\partial t} + \nabla \cdot \mathbf{N} = 0$$

relating some (abstract) charge density N^0/c and current density $\mathbf{N}$ of the same
Gauss' theorem Integrating over a volume V where the current disappears on the surface S we have

$$\int_V \nabla \cdot \mathbf{N} dV = \int_S \mathbf{N} \cdot d\mathbf{S} = 0$$

which gives the conservation law $\frac{d}{dt} Q = 0$ for the **Noether charge**

$$Q = \int_V \frac{N^0}{c} dV$$

The Energy-Momentum Tensor

compare to no
explicit t
dependence in
mechanics

For a Lagrangian $\mathcal{L}$ with no explicit dependence on coordinates x^μ a translation

$$x^\mu \rightarrow x'^\mu = x^\mu + a^\mu$$

is a symmetry transformation (with $\delta F^\mu = 0$).

The corresponding Noether current is

$$N^\mu = -\pi^\mu \partial_\nu \phi a^\nu + \mathcal{L} \delta^\mu_\nu a^\nu \equiv -T^\mu_\nu a^\nu$$

There are then four conserved currents $\partial_\mu T^{\mu\nu} = 0$, where

$$T^{\mu\nu} = T^\mu_\rho g^{\rho\nu} = \pi^\mu \partial^\nu \phi - \mathcal{L} g^{\mu\nu}$$

is called the **canonical energy-momentum tensor**.

The four conserved Noether charges under translation invariance (symmetry) are

$$P^\nu = \frac{1}{c} \int T^{0\nu} dV$$

Here T^{00} is the Hamiltonian density

$$T^{00} = \pi^0 \partial^0 \phi - \mathcal{L} = \pi \partial_0 \phi - \mathcal{L} = \mathcal{H}$$

and the Noether charges can be written

$$\begin{aligned} P^0 &= \frac{1}{c} \int \mathcal{H} dV = \frac{H}{c} \\ P^i &= \frac{1}{c} \int \pi^0 \partial^i \phi dV = -\frac{1}{c} \int \pi \partial_i \phi dV \end{aligned}$$

Since P^μ is a four-vector, and Hamiltonian represents the total energy of the system, the vector components of P^μ must be the (three) momentum of the field

$$\mathbf{P} = -\frac{1}{c} \int \pi \nabla \phi dV$$

P^μ then contains the energy and momentum of the field $P^\mu = (H/c, \mathbf{P})$