

Scalar field theory

For a real scalar field $\phi(x)$ we start from the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{1}{2} \mu^2 \phi^2 \quad (1)$$

The first term is the **kinetic term** while the second term is called the **mass term**. The equation of motion is found from

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\mu^2 \phi, \quad \pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi, \quad \frac{d}{dx^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \square \phi \quad (2)$$

giving the **Klein-Gordon equation**

$$(\square + \mu^2) \phi = 0 \quad (3)$$

We must have $[\mu] = \text{m}^{-1}$. To fix units we can write $\mu = mc/\hbar$

The Klein-Gordon equation is the relativistic QM e.o.m. for a scalar (spin-0) particle with wavefunction $\phi(x)$ and mass m

The energy-momentum tensor for the scalar field is

$$T^{\mu\nu} = \pi^\mu \partial^\nu \phi - \mathcal{L} g^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \mathcal{L} g^{\mu\nu} \quad (4)$$

and the energy (Hamiltonian) density is

$$T^{00} = \mathcal{H} = \partial^0 \phi \partial^0 \phi - \mathcal{L} = \frac{1}{2} [\partial^0 \phi \partial^0 \phi + (\nabla \phi)^2 + \mu^2 \phi^2] = \frac{1}{2} \left[\frac{1}{c^2} \pi^2 + (\nabla \phi)^2 + \mu^2 \phi^2 \right] \quad (5)$$

The Klein-Gordon equation has plane-wave solutions $\phi(x) = e^{\pm i p x / \hbar}$ where p is the four-momentum since

$$\square \phi = \partial_\mu \partial^\mu e^{\pm i p x / \hbar} = \pm \frac{i}{\hbar} p^\mu \partial_\mu e^{\pm i p x / \hbar} = -\frac{p^2}{\hbar^2} e^{\pm i p x / \hbar} = -\frac{m^2 c^2}{\hbar^2} \phi(x) \quad (6)$$

This allows us to build a general solution as the sum of plane wave solutions

$$\phi(x) = \int \frac{d^3 p}{(2\pi\hbar)^3} \frac{\hbar}{2E} (a(p) e^{-i p x / \hbar} + a^*(p) e^{i p x / \hbar}), \quad (7)$$

where $E = \sqrt{p^2 c^2 + m^2 c^4}$ is relativistic energy, and a is chosen so that $\phi(x)$ is real

It is possible to generalise the Klein-Gordon Lagrangian by writing

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) \quad (8)$$

Of particular interest is the so-called **Higgs potential**

$$V(\phi) = -\mu^2 \phi^2 + \lambda \phi^4 \quad (9)$$

Here $\phi_0 = 0$ is not a stable equilibrium solution, while $\phi_v = \sqrt{\mu^2 / 2\lambda}$ is

We can see this by solving

$$\frac{\partial V}{\partial \phi} = -2\mu^2 \phi + 4\lambda \phi^3 = 0 \quad (10)$$

which gives the solutions $\phi_0 = 0$ and $\phi_v = \sqrt{\frac{\mu^2}{2\lambda}}$ as the equilibrium points

The second derivative

$$\frac{\partial^2 V}{\partial \phi^2} = -2\mu^2 + 12\lambda \phi^2 \quad (11)$$

is negative for ϕ_0 (unstable equilibrium) and positive for ϕ_v (stable equilibrium)

Electromagnetism

The Lagrangian for electromagnetism is

$$\mathcal{L} = -\frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - j^\mu A_\mu \quad (12)$$

where $A^\mu(x)$ is the four-potential and $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ the field strength tensor
The first term is the kinetic term and the second is an **interaction term** with an external four-current j^μ

The equations of motion are found from

$$\frac{\partial \mathcal{L}}{\partial A_\rho} = -j^\rho, \quad (13)$$

and

$$\begin{aligned} \pi^{\pi\rho} = \frac{\partial \mathcal{L}}{\partial(\partial_\pi A_\rho)} &= -\frac{1}{4\mu_0} \frac{\partial}{\partial(\partial_\pi A_\rho)} F^{\mu\nu} F_{\mu\nu} \\ &= -\frac{1}{2\mu_0} F^{\mu\nu} \frac{\partial}{\partial(\partial_\pi A_\rho)} (\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &= -\frac{1}{2\mu_0} F^{\mu\nu} (\delta_\mu^\pi \delta_\nu^\rho - \delta_\nu^\pi \delta_\mu^\rho) \\ &= -\frac{1}{2\mu_0} (F^{\pi\rho} - F^{\rho\pi}) = -\frac{1}{\mu_0} F^{\pi\rho} \end{aligned} \quad (14)$$

$F^{\pi\rho} = -F^{\rho\pi}$

which gives

$$\frac{d}{dx^\pi} \frac{\partial \mathcal{L}}{\partial(\partial_\pi A_\rho)} = -\partial_\pi (\partial^\pi A^\rho - \partial^\rho A^\pi) = -\square A^\rho + \partial^\rho (\partial_\pi A^\pi) \quad (15)$$

The equation of motion are then

$$\square A^\rho - \partial^\rho (\partial_\pi A^\pi) - \mu_0 j^\rho = 0 \quad (16)$$

Finally, using Lorenz gauge $\partial_\pi A^\pi = 0$ we arrive at

$$\square A^\rho = \mu_0 j^\rho \quad (17)$$

which we recognise as **Maxwell's equations** in covariant form

For the free-field case with $j^\nu = 0$ the Lagrangian is invariant under translations and the conserved energy-momentum tensor is

$$T^{\mu\nu} = \pi^{\mu\rho} \partial^\nu A_\rho - \mathcal{L} g^{\mu\nu} = -\frac{1}{\mu_0} F^{\mu\rho} \partial^\nu A_\rho - \mathcal{L} g^{\mu\nu} \quad (18)$$

This is not the same expression as derived earlier in the course, nor is it symmetrical in the indices which it should be to represent the physical energy and momentum. To solve this Belinfante introduced a symmetrisation of the tensor by adding a term

$$t^{\mu\nu} \equiv -\frac{1}{\mu_0} \partial_\rho (F^{\mu\rho} A^\nu) = -\frac{1}{\mu_0} F^{\mu\rho} \partial_\rho A^\nu \quad (19)$$

Lorentz gauge
 $\partial_\rho A^\rho = 0$ and
 Maxwell's
 equations
 $\square A^\mu = 0$

where we have used that

$$\partial_\rho F^{\mu\rho} = \partial_\rho \partial^\mu A^\rho - \partial_\rho \partial^\rho A^\mu = \partial^\mu \partial_\rho A^\rho - \square A^\mu = 0 \quad (20)$$

The **symmetric energy-momentum tensor** energy-momentum tensor $\hat{T}^{\mu\nu}$ is then

$$\begin{aligned} \hat{T}^{\mu\nu} = T^{\mu\nu} + t^{\mu\nu} &= -\frac{1}{\mu_0} F^{\mu\rho} \partial^\nu A_\rho - \frac{1}{\mu_0} F^{\mu\rho} \partial_\rho A^\nu - \mathcal{L} g^{\mu\nu} \\ &= -\frac{1}{\mu_0} F^{\mu\rho} (\partial^\nu A_\rho - \partial_\rho A^\nu) + \frac{1}{4\mu_0} F^{\rho\sigma} F_{\rho\sigma} g^{\mu\nu} \\ &= \frac{1}{\mu_0} \left(-F^{\mu\rho} F^\nu{}_\rho + \frac{1}{4} F^{\rho\sigma} F_{\rho\sigma} g^{\mu\nu} \right) \end{aligned} \quad (21)$$

which is the result that we had previously

The addition of $t^{\mu\nu}$ does not change the physical behaviour of the system since

$$\partial_\mu t^{\mu\nu} = -\frac{1}{\mu_0} \partial_\mu \partial_\rho (F^{\mu\rho} A^\nu) = 0 \quad (22)$$

using the anti-symmetry of $F^{\mu\rho}$

Adding $t^{\mu\nu}$ to the Noether current it remains conserved, and the contribution to the Noether charge can be written as a surface term which disappears on the boundary