

The mass matrix

If the constraints on a system are time-independent (and holonomic), then $\mathbf{r}_i$ is a function of only the q_j , so the total time-derivative is

$$\dot{\mathbf{r}}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \dot{q}_j$$

The kinetic energy is then

$$K = \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2} m_i \sum_{jk} \dot{q}_j \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k = \frac{1}{2} \sum_{jk} \dot{q}_j M_{jk}^i \dot{q}_k = \frac{1}{2} \dot{\mathbf{q}}^T M^i \dot{\mathbf{q}}$$

e.g. in Lecture 3 where M^i is the **mass matrix** for particle i with elements

$$M = m \begin{bmatrix} r^2 & 0 \\ 0 & r^2 \sin^2 \theta \end{bmatrix} \quad M_{jk}^i = m_i \frac{\partial \mathbf{r}_i}{\partial q_j} \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}$$

The Hamiltonian

We define the **Hamiltonian** H of a system as

$$H(q, p, t) \equiv \sum_i p_i \dot{q}_i - L = \sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L$$

NB! The variable dependence of $H(q, p, t)$ implies that we have solved $p_i = \frac{\partial L}{\partial \dot{q}_i}$ for q_i and substituted into the expression

The Hamiltonian is the **Legendre transform** of the Lagrangian

When constraints are time-independent (and holonomic), and the potential is velocity independent, $H = K + V$.

Proof: Letting M^i be the mass matrix of particle i we can write

$$L = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2 - V = \frac{1}{2} \sum_{i,j,k} M_{jk}^i \dot{q}_j \dot{q}_k - V(q),$$

note factor of two and this gives

$$\frac{\partial L}{\partial \dot{q}_j} = \sum_{ik} M_{jk}^i(q) \dot{q}_k$$

The Hamiltonian is then

$$\begin{aligned} H &= \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L = \sum_{ijk} M_{jk}^i(q) \dot{q}_k \dot{q}_j - \left(\frac{1}{2} \sum_{ijk} M_{jk}^i(q) \dot{q}_j \dot{q}_k - V(q) \right) \\ &= \frac{1}{2} \sum_{ijk} M_{jk}^i \dot{q}_j \dot{q}_k + V(q) = K + V \quad \square \end{aligned}$$

Legendre transform

Let f be a function of a variable x , $f(x)$. The **Legendre transform** f^* of this

$$f^*(p) \equiv px - f(x),$$

is a function of the derivative of the original function $p = \frac{df}{dx}$.

In order for the Legendre transform to exist $p = \frac{df}{dx}$ must be invertible so we can find $x(p)$ to eliminate x . Sufficient that $f(x)$ is **convex**, meaning that any straight line through any two points on the graph of $f(x)$ lies above the graph.

In practise this is fulfilled if $f''(x) \geq 0$

The Legendre transform is **involution**, meaning it is its own inverse, $f^{**} = f$

the $\dot{q}_i$ play the role of x above

The Hamiltonian is the Legendre transform of the Lagrangian with respect to all the $\dot{q}_i$ transformed to the generalised momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$, giving

$$H(q, p, t) = L^*(q, p, t) = \sum_i p_i \dot{q}_i - L(q, \dot{q}, t)$$

the general test is whether the Hessian of the function is positive semi-definite

Not all Lagrangians have Legendre transforms and thus Hamiltonians, but many do because of the convex shape of the kinetic energy when the mass matrix is diagonal: $K = \frac{1}{2} M_{ii}(q) \dot{q}_i^2$.

Energy conservation

We can write the total time-derivative of L as

$$\frac{dL(q, \dot{q}, t)}{dt} = \sum_i \frac{\partial L}{\partial \dot{q}_i} \underbrace{\frac{d\dot{q}_i}{dt}}_{\dot{q}_i} + \frac{\partial L}{\partial q_i} \underbrace{\frac{dq_i}{dt}}_{\dot{q}_i} + \frac{\partial L}{\partial t}$$

which can be written

$$\frac{dL}{dt} = \sum_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) - \underbrace{\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i}_{=0 \text{ due to Lagrange's eq.}} + \frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial t}$$

This means that

$$\frac{dH}{dt} = \sum_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) - \frac{dL}{dt} = -\frac{\partial L}{\partial t}$$

Assume no *explicit* time dependence in L , i.e. $\frac{\partial L}{\partial t} = 0$, then H is time independent and constant

What symmetry is causing this? The invariance under $t \rightarrow t' = t + a$.

What is conserved? With time-independent constraints and velocity independent potential, the total energy $H = K + V$.

Total time-derivative

Note that $L' = aL + b$ has the same e.o.m. as L since the b cancels in all partial derivatives, and a is a common factor in all terms that can be divided out.

Is there more that we can do to the Lagrangian without changing the e.o.m.?

Look at the transformation

$$L(q, \dot{q}, t) \rightarrow L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt}F(q, t)$$

and *not* $\dot{q}$ where F is a differentiable function of q .

This addition does not change the e.o.m. and can be used to simplify a Lagrangian with a total time derivative (it can be ignored).

Proof: We define

$$G(q, \dot{q}, t) \equiv \frac{d}{dt}F(q, t) = \sum_j \frac{\partial F}{\partial q_j} \dot{q}_j + \frac{\partial F}{\partial t}$$

Lagrange's equation for L' is

$$\frac{d}{dt} \left(\frac{\partial L'}{\partial \dot{q}_i} \right) - \frac{\partial L'}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} + \underbrace{\frac{d}{dt} \left(\frac{\partial g}{\partial \dot{q}_i} \right) - \frac{\partial g}{\partial q_i}}_{\text{if } = 0 \text{ we are done}}$$

Now

$$\frac{\partial G}{\partial q_i} = \sum_j \frac{\partial}{\partial q_i} \frac{\partial F}{\partial q_j} \dot{q}_j + \frac{\partial}{\partial q_i} \frac{\partial F}{\partial t}$$

and

$$\frac{d}{dt} \left(\frac{\partial G}{\partial \dot{q}_i} \right) = \frac{d}{dt} \left(\frac{\partial F}{\partial q_i} \right) = \sum_j \frac{\partial}{\partial q_j} \frac{\partial F}{\partial q_i} \dot{q}_j + \frac{\partial}{\partial t} \frac{\partial F}{\partial q_i}$$

**Schwarz's
theorem:** f has
continuous
second partial
derivatives

Provided f is well behaved

$$\frac{\partial G}{\partial q_i} = \frac{d}{dt} \left(\frac{\partial G}{\partial \dot{q}_i} \right) \quad \square$$

Velocity dependent potentials

The e.o.m. in the proof of Lagrange's equation was

$$\frac{d}{dt} \left(\frac{\partial K}{\partial \dot{q}_i} \right) - \frac{\partial K}{\partial q_i} = \mathcal{F}_i$$

We found Lagrange's equation for a velocity independent potential (conservative force) $V(q, t)$ where the generalised force was

$$\mathcal{F}_i = -\frac{\partial V}{\partial q_i}$$

For some generalised forces we can find a velocity dependent potential $U(q, \dot{q}, t)$, called **monogenic** potential, that fulfils

$$\mathcal{F}_i = \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_i} \right) - \frac{\partial U}{\partial q_i}$$

These then lead to Lagrange's equation by *defining* the Lagrangian as

$$L(q, \dot{q}, t) = K(q, \dot{q}, t) - U(q, \dot{q}, t)$$