

Potential for charged particle in electromagnetic field

cannot write
potential
without velocity
 $\mathbf{r} = (x_1, x_2, x_3)$

Particle of charge q and mass m , with velocity $\mathbf{v}$ in an electric $\mathbf{E}$ and magnetic field $\mathbf{B}$ feels a force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Since there are no constraints, choose Cartesian $q_i = x_i$ and $\dot{q}_i = v_i$. The generalised force on the particle is then just the ordinary force components

$$\mathcal{F}_i = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial q_i} = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial x_i} = F_i$$

We write the e.m. fields using the **scalar** $\phi(\mathbf{r}, t)$ and **vector** $\mathbf{A}(\mathbf{r}, t)$ **potentials** as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}$$

see Sec. 9.5 in
book

We will later show that $\mathbf{E}$ & $\mathbf{B}$ can always be represented this way
The force is then

$$\mathbf{F} = q \left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A}) \right) = q \left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A} \right)$$

where we have used

see Rottmann
p. 64

$$\mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - \underbrace{\mathbf{A} \times (\nabla \times \mathbf{v})}_{=0} - \underbrace{(\mathbf{A} \cdot \nabla)\mathbf{v}}_{=0} - (\mathbf{v} \cdot \nabla)\mathbf{A}$$

The total time derivative of $\mathbf{A}$ is

$$\frac{d\mathbf{A}}{dt} = \sum_j \frac{\partial \mathbf{A}}{\partial x_j} \frac{dx_j}{dt} + \frac{\partial \mathbf{A}}{\partial t} = \sum_j \frac{\partial \mathbf{A}}{\partial x_j} v_j + \frac{\partial \mathbf{A}}{\partial t} = (\mathbf{v} \cdot \nabla)\mathbf{A} + \frac{\partial \mathbf{A}}{\partial t},$$

so the generalised force can be written

$$\mathcal{F}_i = \frac{d}{dt}(-qA_i) - \frac{\partial}{\partial x_i}(q\phi - q\mathbf{v} \cdot \mathbf{A})$$

We see the condition for a monogenic potential is fulfilled by $U(\mathbf{r}, \mathbf{v}, t) = q\phi - q\mathbf{v} \cdot \mathbf{A}$.
The Lagrangian for a charged particle in an e.m. field is then

$$L = K - U = \frac{1}{2}m\mathbf{v}^2 - q\phi(\mathbf{r}, t) + q\mathbf{v} \cdot \mathbf{A}(\mathbf{r}, t).$$

Hamiltonian for e.m. force

The generalised momentum is

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial v_i} = mv_i + qA_i, \quad \mathbf{p} = m\mathbf{v} + q\mathbf{A}$$

some literature
switches this
using π_i for
generalised
momentum

compared to a free particle with $L = \frac{1}{2}m\mathbf{v}^2$ and $p_i = \frac{\partial L}{\partial v_i} = mv_i$.
To differentiate $\pi_i = mv_i$ is often called **mechanical momentum**.

Notice here how we can solve p_i for the velocity v_i

$$\mathbf{v} = \frac{1}{m}(\mathbf{p} - q\mathbf{A})$$

The expression for the Hamiltonian is

$$H \equiv \sum_i p_i \dot{q}_i - L = \mathbf{p} \cdot \mathbf{v} - \frac{1}{2}m\mathbf{v}^2 + q\phi - q\mathbf{v} \cdot \mathbf{A} = \frac{1}{2}m\mathbf{v}^2 + q\phi$$

We see that $H \neq K + U$ but $H = K + V_E$ where $V_E = q\phi$ is the potential energy of the change in an electric potential.

Substituting out $\mathbf{v}$ we can write the Hamiltonian in terms of the momentum

$$H(\mathbf{r}, \mathbf{p}, t) = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi$$

H is a constant of motion if L is explicitly time-independent, i.e. if ϕ and $\mathbf{A}$ are. If so

electric force: $\mathbf{F}_e = -q\nabla\phi$ conservative force

magnetic force: $\mathbf{F}_m = q\mathbf{v} \times \mathbf{B}$ perpendicular to motion. no work (no change in energy)

Gauge invariance (internal symmetry)

If we modify e.m. potentials

$$\phi \rightarrow \phi' = \phi - \frac{\partial\chi}{\partial t}, \quad \mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \nabla\chi,$$

Schwarz'

theorem:

continuous
second partial
derivatives

where $\chi(\mathbf{r}, t)$ is a well behaved function, then

$$\mathbf{E}' = -\nabla \left(\phi - \frac{\partial\chi}{\partial t} \right) - \frac{\partial}{\partial t}(\mathbf{A} + \nabla\chi) = \mathbf{E} + \nabla \frac{\partial\chi}{\partial t} - \frac{\partial}{\partial t} \nabla\chi = \mathbf{E}$$

$$\mathbf{B}' = \nabla \times (\mathbf{A} + \nabla\chi) = \mathbf{B} + \nabla \times \nabla\chi = \mathbf{B}$$

using that $\nabla \times (\nabla\chi) = 0$ (Rottmann p. 64)

This freedom to change potentials ϕ and $\mathbf{A}$ but not $\mathbf{E}$ and $\mathbf{B}$ is called **gauge invariance**

What about the Lagrangian?

$$L \rightarrow L' = L + q \frac{\partial\chi}{\partial t} + q\mathbf{v} \cdot \nabla\chi = L + q \left(\frac{\partial\chi}{\partial t} + \sum_i \frac{\partial\chi}{\partial x_i} v_i \right) = L + q \frac{d\chi}{dt}$$

Total time derivative $\Rightarrow$ term can be ignored.

No change in e.o.m! **Internal gauge symmetry.**

What is the conserved quantity? Electric charge q

we have to wait
to see this

Note that the generalised momentum and the Hamiltonian does change

$$p'_i \equiv \frac{\partial L'}{\partial v_i} = \frac{\partial L}{\partial v_i} + q \frac{\partial}{\partial v_i} \left(\frac{\partial \chi}{\partial t} + \sum_j \frac{\partial \chi}{\partial x_j} v_j \right) = p_i + q \frac{\partial \chi}{\partial x_i}, \quad \mathbf{p}' = \mathbf{p} + q \nabla \chi$$

in QM can be
compensated by
 $\psi \rightarrow \psi' =$
 $e^{iq\chi(\mathbf{r},t)}\psi$

and

$$H' = \frac{1}{2m}(\mathbf{p}' - q\mathbf{A}')^2 + q\phi' = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + q\phi - q \frac{\partial \chi}{\partial t} = H - q \frac{\partial \chi}{\partial t}$$

Hamilton's e.o.m.

As a function of q, p & t , the infinitesimal dH of $H(q, p, t)$ is

total time
derivative

$$dH = \sum_i \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right) + \frac{\partial H}{\partial t} dt \quad (1)$$

using product
rule, and
multiplying by
 dt

Compare this to using $H(q, \dot{q}, t) = \sum p_i \dot{q}_i - L(q, \dot{q}, t)$

$$\begin{aligned} dH &= \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i) - dL \\ &= \sum_i [\dot{q}_i dp_i + p_i d\dot{q}_i - \underbrace{\frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i}_{\text{using total time-derivative of } L} - \frac{\partial L}{\partial t} dt] \\ &= \sum_i \left(\dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i \right) - \frac{\partial L}{\partial t} dt \end{aligned} \quad (2)$$

these are purely
mathematical
relationships
between L and
 H

Comparing (1) & (2) we see that

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i}, \quad \text{and} \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Now, Lagrange's equation can be written

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \quad \Leftrightarrow \quad \frac{d}{dt} p_i = \dot{p}_i = \frac{\partial L}{\partial q_i}$$

Thus we find the e.o.m.

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

can also be
shown in reverse
direction

which are **Hamilton's equations**.

These are two (coupled) differential equations, but first order

The third equation tells us that the Hamiltonian is only time-dependent if it is explicitly time-dependent

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

WARNING: The first equality only holds for paths $q(t)$ that were solutions to the e.o.m., it is not a generic relationship between H and L .

Example: Charged particle in constant magnetic field

$\mathbf{E} = 0$ Trick to get (only) constant magnetic field: choose $\phi = 0$ and $\mathbf{A} = -\frac{1}{2}\mathbf{r} \times \mathbf{B}_0$:

$$\begin{aligned}\mathbf{B} &= \nabla \times \mathbf{A} = -\frac{1}{2}\nabla \times (\mathbf{r} \times \mathbf{B}_0) \\ &= -\frac{1}{2}[(\mathbf{B}_0 \cdot \nabla)\mathbf{r} - (\mathbf{r} \cdot \underbrace{\nabla}_{=0})\mathbf{B}_0 + \mathbf{r}(\underbrace{\nabla \cdot \mathbf{B}_0}_{=0}) - \mathbf{B}_0(\underbrace{\nabla \cdot \mathbf{r}}_{\frac{\partial}{\partial x_i}x_i=3})] \\ &= -\frac{1}{2}[\mathbf{B}_0 - 3\mathbf{B}_0] = \mathbf{B}_0\end{aligned}$$

Where we have used Rottmann p. 64:

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a})$$

and

$$(\mathbf{B}_0 \cdot \nabla)\mathbf{r} = \left(\sum_i B_{0i} \frac{\partial}{\partial x_i}\right)(x_1, x_2, x_3) = (B_{01}, B_{02}, B_{03}) = \mathbf{B}_0$$

The components of the vector potential are:

$$A_x = -\frac{1}{2}yB_0, \quad A_y = +\frac{1}{2}xB_0, \quad A_z = 0$$

and the Hamiltonian is then

$$H = \frac{1}{2m}(\mathbf{p} - q\mathbf{A})^2 + e\phi = \frac{1}{2m} \left[\left(p_x + \frac{1}{2}qB_0y \right)^2 + \left(p_y - \frac{1}{2}qB_0x \right)^2 + p_z^2 \right]$$

$\mathbf{p} = m\mathbf{v} + q\mathbf{A}$ H is (explicitly) independent of t so $H = \frac{1}{2}m\mathbf{v}^2 = K$ is conserved. Magnetic field only changes direction of motion.
if q_i is absent in L it is also absent in H and z is a cyclic coordinate. From Hamilton's equations
vice versa

$$\dot{p}_z = -\frac{\partial H}{\partial z} = 0$$

so as expected the generalised momentum p_z is conserved.

We find p_z :

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A} \quad p_z = m\dot{z} + eA_z = m\dot{z}$$

so p_z is the same as the mechanical momentum and the particle has constant velocity $\dot{z}$ along magnetic field with e.o.m.

$$z = z_0 + \dot{z}t$$

Perpendicular to magnetic field we have

four equations
in x, y, p_x, p_y ,
could solve
numerically here

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p_x} = \frac{1}{m} \left(p_x + \frac{1}{2}qB_0y \right) \quad (1), & \dot{p}_x &= -\frac{\partial H}{\partial x} = \frac{qB_0}{2m} \left(p_y - \frac{1}{2}qB_0x \right) \quad (2), \\ \dot{y} &= \frac{\partial H}{\partial p_y} = \frac{1}{m} \left(p_y - \frac{1}{2}qB_0x \right) \quad (3), & \dot{p}_y &= -\frac{\partial H}{\partial y} = -\frac{qB_0}{2m} \left(p_x + \frac{1}{2}qB_0y \right) \quad (4),\end{aligned}$$

Comparing (2) $-\frac{1}{2}qB_0(3)$, and (4) $-\frac{1}{2}qB_0(1)$ gives

$$\dot{p}_x - \frac{1}{2}qB_0\dot{y} = 0 \quad \text{and} \quad \dot{p}_y + \frac{1}{2}qB_0\dot{x} = 0$$

which means that

exercise:
show that $\mathbf{K} = m\mathbf{v} - q\mathbf{r} \times \mathbf{B}_0$

$$K_x = p_x - \frac{1}{2}qB_0y \quad \text{and} \quad K_y = p_y + \frac{1}{2}qB_0x$$

are constants of motion $\left(\frac{dK_x}{dt} = \frac{dK_y}{dt} = 0\right)$

Combining (2) $+\frac{1}{2}qB_0(3)$ and (4) $-\frac{1}{2}qB_0(1)$ gives

$$\dot{p}_x + \frac{1}{2}qB_0\dot{y} = \frac{qB_0}{m} \left(p_y - \frac{1}{2}qB_0x\right) \quad (5)$$

$$\dot{p}_y - \frac{1}{2}qB_0\dot{x} = -\frac{qB_0}{m} \left(p_x + \frac{1}{2}qB_0y\right) \quad (6)$$

Since $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$, the mechanical momentum $\boldsymbol{\pi}$ is

$$\pi_i \equiv mv_i = p_i - qA_i \quad \Rightarrow \quad \pi_x = p_x + \frac{1}{2}qB_0y, \quad \pi_y = p_y - \frac{1}{2}qB_0x$$

Thus (5) & (6) can be written as

$$\dot{\pi}_x = \frac{qB_0}{m}\pi_y, \quad \text{and} \quad \dot{\pi}_y = -\frac{qB_0}{m}\pi_x$$

which means each mechanical momentum satisfies

HO: $\ddot{x} = -\omega^2x$

$$\ddot{\pi}_x = -\frac{q^2B_0^2}{m^2}\pi_x, \quad \text{and} \quad \ddot{\pi}_y = -\frac{q^2B_0^2}{m^2}\pi_y$$

or

$$\ddot{\pi}_x + \omega^2\pi_x = 0, \quad \text{and} \quad \ddot{\pi}_y + \omega^2\pi_y = 0$$

with $\omega = \frac{qB_0}{m}$ the circular frequency of a HO, here called **cyclotron frequency**.

initial momentum Choose the two first initial conditions as $\pi_x = \pi_0$ and $\pi_y = 0$ at $t = 0$.
We have solutions for the momenta

$$\pi_x = \pi_0 \cos \omega t, \quad \text{and} \quad \pi_y = \pi_0 \sin \omega t.$$

Solving $K_x = p_x - \frac{1}{2}qB_0y$ and $\pi_x = \pi_0 \cos \omega t = p_x + \frac{1}{2}qB_0y$ for y gives

$$y = \frac{1}{qB_0} (\pi_0 \cos \omega t - K_x)$$

or

$$y = y_0 + R \cos \omega t, \quad \text{with} \quad y_0 = -\frac{K_x}{qB_0} \quad \text{and} \quad R = \frac{\pi_0}{qB_0}$$

Likewise $x = x_0 + R \sin \omega t$ with $x_0 = \frac{K_y}{qB_0}$, where K_x and K_y fixes initial position.
The motion in the xy -plane is a circle of radius R around (x_0, y_0) , in total a spiral.