

Poisson brackets

nothing to do
with fish

We define the **Poisson bracket** for two functions $f(q, p, t)$ and $g(q, p, t)$ that have continuous double derivatives as

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

Such functions are called **dynamic variables**.

It is easy to see that the following properties hold ($a, b \in \mathbb{R}$)

$$\{f, g\} = -\{g, f\}, \quad \{f, f\} = 0, \quad \{af + bg, h\} = a\{f, h\} + b\{g, h\}$$

where $h(q, p, t)$ is another dynamic variable.

It harder to prove that, but still true, that

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \quad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

This notation is particularly useful for writing the total time derivative of a function

$$\begin{aligned} \frac{df}{dt} &= \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t} \\ &= \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t} \quad (\text{using Hamilton's equations}) \end{aligned}$$

So

$$\dot{f} = \{f, H\} + \frac{\partial f}{\partial t} \tag{1}$$

All f where $\{f, H\} = 0$ and without explicit time-dependence are constants of motion.

Setting $f = p_i$ or $f = q_i$ into (1) gives us Hamilton's equations

$$\dot{p}_i = \{p_i, H\}, \quad \dot{q}_i = \{q_i, H\}$$

sometimes
called the
**fundamental
Poisson
brackets**

The Poisson brackets of the generalised coordinates and momenta are:

$$\{q_i, p_j\} = \sum_k \frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_i} \frac{\partial p_j}{\partial q_j} = \delta_{ik} \delta_{jk} = \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0$$

Heisenberg
picture:
operators
change with
time, not states

In QM the **Heisenberg equation** for the time dependence of an operator $\hat{f}(\hat{q}, \hat{p}, t)$ is

$$\dot{\hat{f}} = \frac{1}{i\hbar} [\hat{f}, \hat{H}] + \frac{\partial \hat{f}}{\partial t}$$

Comparing to classical mechanics we often say that the Poisson brackets are “promoted” to commutators in QM. The relationships

$$\hat{f}(\hat{q}, \hat{p}, t) \leftrightarrow f(q, p, t), \quad \text{and} \quad \frac{1}{i\hbar} [\hat{f}, \hat{g}] \leftrightarrow \{f, g\}$$

is often called the **correspondence principle**.

More formally we can show that in the classical limit $\hbar \rightarrow 0$

$$\lim_{\hbar \rightarrow 0} \left\langle \frac{1}{i\hbar} [\hat{f}, \hat{g}] \right\rangle = \{f, g\}$$

We see that this is consistent with the defining commutator

$$[\hat{q}_i, \hat{p}_j] = i\hbar \delta_{ij}$$

This means that the e.o.m. of QM reduce to Hamilton's equations in the classical limit, for example

$$\dot{\hat{p}} = \frac{d}{dt} \hat{p} = \frac{1}{i\hbar} [\hat{p}, \hat{H}]$$

gives $\dot{p} = \{p, H\}$

Example: Harmonic oscillator (HO) in 1D

$$K = \frac{1}{2} m \dot{x}^2, \quad V = \frac{1}{2} k x^2, \quad L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$$

Generalised momentum

$$p = \frac{\partial L}{\partial \dot{x}} = m \dot{x} \Rightarrow \dot{x} = \frac{p}{m}$$

Hamiltonian in x and p

$$H(x, p) = p \dot{x} - L = \frac{p^2}{m} - \frac{p^2}{2m} + \frac{1}{2} k x^2 = \frac{p^2}{2m} + \frac{1}{2} k x^2$$

no constraints, velocity independent potential

Since $\frac{dH}{dt} = -\frac{\partial H}{\partial t} = 0$ total energy $H = K + V$ is conserved for solutions

Hamilton's equations are

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -kx$$

These can now be solved numerically

We can check that we get ordinary HO equation back by eliminating p using the first equation, $\dot{p} = m\ddot{x}$, so $\ddot{x} = -\frac{k}{m}x$

Example: Pendulum

A pendulum with mass m and length ℓ has (Lecture 1)

$$L(\theta, \dot{\theta}) = K - V = \frac{1}{2} m \ell^2 \dot{\theta}^2 + mg\ell \cos \theta, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m \ell^2 \dot{\theta}$$

The Hamiltonian is

$$H(\theta, p_\theta) = p_\theta \dot{\theta} - L = \frac{p_\theta^2}{m \ell^2} - \frac{p_\theta^2}{2m \ell^2} - mg\ell \cos \theta = \frac{p_\theta^2}{2m \ell^2} - mg\ell \cos \theta$$

velocity	Explicit time independence gives conservation of energy $H = K + V$
independent	
potential & time	Hamilton's equations are
independent	
constraints	

$$\dot{\theta} = \frac{\partial H}{\partial p_{\theta}} = \frac{p_{\theta}}{m\ell^2}$$

$$\dot{p}_{\theta} = -\frac{\partial H}{\partial \theta} = -mg\ell \sin \theta$$

Can be solved numerically or analytically as HO in limit $\theta \ll 1$ ($\sin \theta \simeq \theta$)