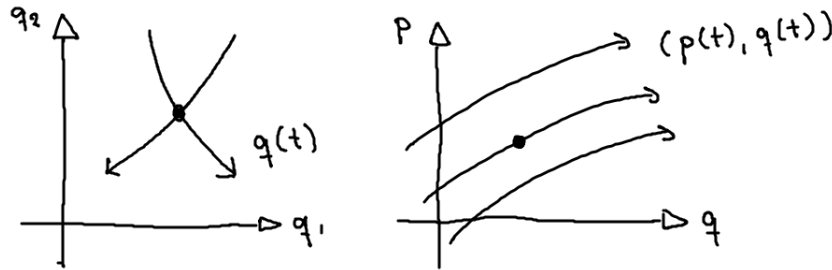


Lagrangian versus Hamiltonian

Lagrangian $L(q, \dot{q}, t) \Rightarrow$	description in terms of q_i and $\dot{q}_i$ e.o.m. second order diff. equation easy to write down has theoretical advantages (in particular in QFT)
Hamiltonian $H(q, p, t) \Rightarrow$	description in terms of q_i and p_i e.o.m. coupled first order diff. equations need to solve expression for p_i in terms of $\dot{q}_i = \dot{q}_i(q, p, t)$ easy to find numerical solutions we can use phase space appears in QM

Phase space



Configuration space: d -dimensional $q = \{q_1, \dots, q_d\}$.

The extension to $2d$ -dimensional space $\{q, \dot{q}\}$ or $\{q, p\}$ is called **phase space**.

We need $2d$ initial data to give a unique trajectory (d second order equations or $2d$ first order)

Through a point in configuration space there are many trajectories (solutions), while there is (almost always) a unique trajectory in phase space.

Crossings in phase space correspond to unstable equilibria.

Useful for a qualitative understanding – but we must be careful with relationship to *physical* motion

Example: 1D harmonic oscillator

Hamiltonian

$$H(x, p) = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

H is
Hamiltonian
function, E is a
particular value

Total energy $H = K + V = E$ is conserved so $p^2 + kmx^2 = 2mE$ forms ellipse in px -plane

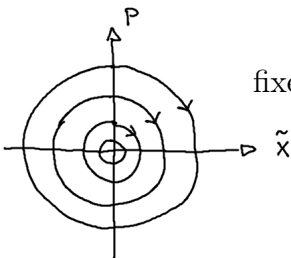
Rescale $\tilde{x} = \sqrt{km}x$ and $p^2 + \tilde{x}^2 = 2mE$ is a circle with radius $\sqrt{2mE}$

We can view H as a **phase space potential** where the trajectory is orthogonal to the gradient (isocurve)

Hamilton's equations

$$\dot{x} = \frac{p}{m}, \quad \dot{p} = -kx$$

fixes direction and shows motion is clockwise in phase space



Example: Pendulum

Hamiltonian

$$H(\theta, p_\theta) = \frac{p_\theta^2}{2m\ell^2} - mg\ell \cos \theta$$

Alternatively, moving $V = 0$ to $\theta = 0$

$$\cos \theta = 1 - 2 \sin^2 \frac{\theta}{2} \quad H(\theta, p_\theta) = \frac{p_\theta^2}{2m\ell^2} + 2mg\ell \sin^2 \frac{\theta}{2}$$

Conservation of energy $H = K + V = E$ gives

$$p_\theta^2 + 4m^2 g \ell^3 \sin^2 \frac{\theta}{2} = 2m\ell^2 E$$

Rescaling to unitless momentum $\tilde{p} = \frac{p_\theta}{m\sqrt{g\ell^3}}$ gives

$$\tilde{p}^2 + 4 \sin^2 \frac{\theta}{2} = \frac{2E}{mg\ell}$$

Hamilton's equations for the system fixes direction of motion (see slide)

$$\left[\sqrt{\frac{g}{\ell}}\right] = s^{-1} \quad \dot{\theta} = \frac{p_\theta}{m\ell^2} = \sqrt{\frac{g}{\ell}} \tilde{p}$$
$$\dot{p}_\theta = -mg\ell \sin \theta \quad \text{or} \quad \dot{\tilde{p}} = -\sqrt{\frac{g}{\ell}} \sin \theta$$

Unstable equilibria for $\theta = (2n + 1)\pi$, $n = 0, 1, \dots$

Non-Hamiltonian systems

System with no Hamiltonian description, e.g. due to friction.

A system described by $\dot{p}(p, q, t)$ and $\dot{q}(p, q, t)$ is a **dynamical system**.

We can still look at phase space by solving the differential equations.

If for some values of the q_i and p_i , $\dot{p}_i = 0$ and $\dot{q}_i = 0$ for all i , the system has a **fixed point** where it will be stuck. This is an equilibrium point of the system.

As an example, a damped pendulum does not have a Hamiltonian description but the e.o.m. (Hamilton's equations) are

$$\dot{\theta} = \frac{p_\theta}{m\ell^2}, \quad \dot{p}_\theta = -\lambda \sqrt{\frac{g}{\ell}} p_\theta - mg\ell \sin \theta$$

where λ is a damping parameter. In the unitless momentum $\tilde{p}$ this is

$$\dot{\theta} = \sqrt{\frac{g}{\ell}} \tilde{p}, \quad \dot{\tilde{p}} = -\sqrt{\frac{g}{\ell}} (\lambda \tilde{p} + \sin \theta)$$

This can be plotted in phase space (see slide).

Fixed points for $\tilde{p} = 0$ and $\theta = 2\pi n$, $n = 0, 1, 2, \dots$

The action

A **functional** is a “function of a function”, e.g. the functional I given by

$$I[f(x)] = \int_a^b f(x) dx$$

takes a function $f(x)$ as input and evaluates to a value in $\mathbb{R}$ (or $\mathbb{C}$).

We define the **action** as the functional S of the path $q(t)$ given by

$$S[q(t)] = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t) dt,$$

$[S] = [\hbar] = \text{Js}$ where L is the Lagrangian, and t_1 and t_2 is the initial and final time of the motion.

Hamilton's principle

or $q_i(t_1)$ and $p_i(t_1)$ in the Hamiltonian formulation

We can specify initial conditions for the e.o.m. at t_1 by the $2d$ -values $q_i(t_1)$ and $\dot{q}_i(t_1)$, (**Cauchy boundary condition**)

The Lagrange equation will propagate the solution to t .

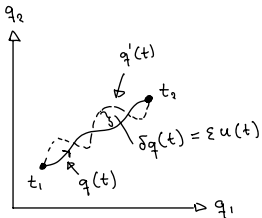
Alternatively, we can specify $q_i(t_1)$ and $q_i(t_2)$ (**Dirichlet boundary condition**)

What is special about trajectories $q_i(t)$ that fulfil e.o.m. and initial conditions?

Hamilton's principle says that for a path $q_i(t)$ which fulfils the e.o.m. (thus is a solution) the action S is **stationary**

This means that under a small variation of the path $q_i(t)$

$$q_i(t) \rightarrow q'_i(t) = q_i(t) + \delta q_i(t),$$



where we keep the path unchanged at the endpoints $\delta q_i(t_1) = \delta q_i(t_2) = 0$, the action is unchanged

$$\delta S = S' - S = 0$$

to first order in δq_i

This can be seen as requiring the derivative of the action with respect to the path to be zero, and can be written in terms of **functional derivatives**

$$\frac{\delta S[q(t)]}{\delta q_i} = 0$$

If we want to do this more formally we can parameterise the change in q by

$$q_i(t) \rightarrow q'_i(t) = q_i(t) + \epsilon u_i(t)$$

where u is some function of t , $u_i(t_1) = u_i(t_2) = 0$, and ϵ a parameter that controls the size of the variation of the path.

WARNING: in general it does not have to be a minimum

This is also called the **principle of least action** because we find an extremum of S .

Equivalence to Lagrange's equation

Demanding $\delta S = 0$ is equivalent to fulfilling Lagrange's equations

Proof: With the transformation $q_i \rightarrow q'_i = q_i + \delta q_i$ the Lagrangian changes

$$L \rightarrow L' = L + \delta L$$

Then

$$\begin{aligned} \delta S &= S' - S = \int_{t_1}^{t_2} L' dt - \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} \delta L(q(t), \dot{q}(t), t) dt \\ &= \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt \end{aligned}$$

where we look at the variation in L for a fixed time $\delta t = 0$.

We can rewrite this using the product rule to remove $\delta \dot{q}_i = \frac{d}{dt} \delta q_i(t)$.

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right) dt \\ &= \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt + \underbrace{\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_1}^{t_2}}_{=0 \text{ because } \delta q_i(t_1) = \delta q_i(t_2) = 0} \\ &= \int_{t_1}^{t_2} \sum_i \underbrace{\left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right)}_{\text{Lagrange's equation}} \delta q_i dt \end{aligned} \tag{1}$$

This gives $\delta S = 0$ if all the $q_i(t)$ fulfil Lagrange's equation, and if $\delta S = 0$ for any δq_i then Lagrange equation must be fulfilled.

The minimisation of S only uses the mathematical properties of L (what it depends on). This technique is called the **calculus of variations** (*variasjonsregning*) and can be used to solve many problems

Functional derivative (not examinable)

Formally the **functional derivative** of a functional $F[\phi]$ at the function $\phi = \bar{\phi}$ is denoted

$$\left. \frac{\delta F[\phi]}{\delta \phi(x)} \right|_{\phi=\bar{\phi}}$$

and is given from

$$\int_D dx \left. \frac{\delta F[\phi]}{\delta \phi(x)} \right|_{\phi=\bar{\phi}} u(x) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [F[\bar{\phi} + \epsilon u] - F[\bar{\phi}]]$$

if the limit exists and $\frac{\delta F[\phi]}{\delta \phi(x)}$ is independent of $u(x)$. Here, D is the domain that the functions $\phi(x)$ are defined on.

The functional derivative is a distribution in D

One can show that the derivative of the action w.r.t. the path is

$$\frac{\delta S[q]}{\delta q_i(t)} = \frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}$$

which implies the equivalence between Hamilton's principle and Lagrange's equation
Proof:

$$\begin{aligned} \int dt \sum_i \frac{\delta S[q]}{\delta q_i(t)} u_i(t) &= \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} (S[q(t) + \epsilon u(t)] - S[q(t)]) \\ &= \int_{t_1}^{t_2} dt \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [L[q(t) + \epsilon u(t), \dot{q} + \epsilon \dot{u}, t] - L[q, \dot{q}, t]] \\ &= \int_{t_1}^{t_2} dt \sum_i \left(\frac{\partial L}{\partial q_i} u_i + \frac{\partial L}{\partial \dot{q}_i} \dot{u}_i \right) \\ &= \int_{t_1}^{t_2} dt \sum_i \left[\frac{\partial L}{\partial q_i} u_i - \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) u_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} u_i \right) \right] \\ &= \int_{t_1}^{t_2} dt \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) u_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} u_i \Big|_{t_1}^{t_2} \\ &= \int_{t_1}^{t_2} dt \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) u_i \end{aligned}$$

Calculus of variations

An example of use outside of mechanics. What is the shortest distance between two points (in the plane): (x_1, y_1) and (x_2, y_2) ?

Infinitesimal distance:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx} \right)^2} dx = \sqrt{1 + \dot{y}^2} dx$$

where we define $\dot{y} \equiv \frac{dy}{dx}$.

Total distance:

$$D[y(x)] = \int_{x_1}^{x_2} ds = \int_{x_1}^{x_2} \sqrt{1 + \dot{y}^2} dx = \int_{x_1}^{x_2} f(y, \dot{y}, x) dx$$

where $f(y, \dot{y}, x) = \sqrt{1 + \dot{y}^2}$.

Notice how this is now formulated just like the action.

When is D minimal? D has an extremum when $\delta D = 0$ under variations of $y(x)$

We can take over mathematical result in (1) by identifying

$$\begin{aligned} L &\rightarrow f \\ q &\rightarrow y \\ \dot{q} &\rightarrow \dot{y} \\ t &\rightarrow x \end{aligned}$$

This immediately gives the equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial \dot{y}} \right) = 0$$

With

$$\frac{\partial f}{\partial y} = 0, \quad \frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}}$$

we get

$$\frac{d}{dx} \left(\frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} \right) = 0, \quad \text{or} \quad \frac{\dot{y}}{\sqrt{1 + \dot{y}^2}} = c$$

where c is a constant. We find $\dot{y}$

$$\begin{aligned} \dot{y}^2 &= c^2 (1 + \dot{y}^2) \\ (1 - c^2) \dot{y}^2 &= c^2 \\ \dot{y} &= \sqrt{\frac{c^2}{1 - c^2}} = a \end{aligned}$$

The only solution is $\dot{y} = a$ (another constant), but this means $y(x) = ax + b$

Can be generalised to other ds in other geometries, e.g. shortest path on a sphere.