

Noether's theorem

Proved in 1915 by Emmy Noether.

Noether's theorem essentially says:

Continuous symmetry $\Rightarrow$ conserved quantity

A **continuous symmetry** is a transformation with a continuous parameter s , e.g. $s \in \mathbb{R}$, that leaves something unchanged

The mathematical language of continuous symmetries is Lie groups

As symmetry we will use *(form) symmetries of the equations of motion*

A general transformation of the coordinates is given by

$$t \rightarrow t'(q, t), \quad q_i \rightarrow q'_i(q, t)$$

We assume these are invertible, i.e. that $q(q', t')$ and $t(q', t')$ can be found

The e.o.m. (Lagrange equations) will be form invariant under this transformations if

$\ddot{q}_i = f_i[q, \dot{q}, t] \rightarrow$
 $\ddot{q}'_i = f_i[q', \dot{q}', t']$
 where the
 functions f_i are
 the same

$$L[q'(q, t), \dot{q}'(q, \dot{q}, t), t'(q, t)] \frac{dt'}{dt} = aL[q, \dot{q}, t] + \underbrace{\frac{dF(q, t)}{dt}}_{\text{gauge function}}$$

NB! Not all
 form symmetries
 of the e.o.m.
 have Lagrangian
 symmetries

where $F(q, t)$ is some function of the q_i and t only and a some constant.

We sometime say that if this is the case we have a symmetry of the Lagrangian

This leads to a symmetry of the action

$$S[q(t)] \rightarrow S'[q'(t')] = S[q'(t')]$$

Proof of Noether's theorem

Many ways to do this with varying degree of mathematical formality and correctness

Assume we have infinitesimal transformations

not the same
 transformation
 as used in
 minimisation of
 action which
 keeps endpoints
 fixed

$$t \rightarrow t' = t + \delta t(q, t), \quad q_i \rightarrow q'_i = q_i + \delta q_i(q, t)$$

If these are a symmetry of Lagrangian leading to form symmetry for e.o.m. we have

$$L[q'(q, t), \dot{q}'(q, \dot{q}, t), t'(q, t)] \frac{dt'}{dt} = L[q, \dot{q}, t] + \frac{dF(q, t)}{dt} \quad (1)$$

We can write the left-hand side of (1) using to first order in infinitesimals

$$L[q'(q, t), \dot{q}'(q, \dot{q}, t), t'(q, t)] = L[q, \dot{q}, t] + \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) + \frac{\partial L}{\partial t} \delta t$$

and

$$\frac{dt'}{dt} = 1 + \frac{d}{dt} \delta t$$

Multiplying these, keeping only terms to first order in infinitesimals, we get

$$L + \sum_i \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) + \frac{\partial L}{\partial t} \delta t + L \frac{d}{dt} \delta t$$

velocity can
change both
because position
coordinates
change and time
changes

simplifying notation using $L = L[q, \dot{q}, t]$ since all Lagrangians have same arguments
We also have

$$\begin{aligned}\delta\dot{q}_i &\equiv \dot{q}'_i - \dot{q}_i = \frac{dt}{dt'} \frac{d}{dt} (q_i + \delta q_i) - \dot{q}_i = \frac{dt}{dt'} (\dot{q}_i + \frac{d}{dt} \delta q_i) - \dot{q}_i \\ &= (1 - \frac{d}{dt'} \delta t) (\dot{q}_i + \frac{d}{dt} \delta q_i) - \dot{q}_i = \dot{q}_i + \frac{d}{dt} \delta q_i - \dot{q}_i \frac{d}{dt} \delta t - \dot{q}_i \\ &= \frac{d}{dt} \delta q_i - \dot{q}_i \frac{d}{dt} \delta t\end{aligned}$$

again keeping only terms to first order in infinitesimals.

Comparing to the right-hand side of (1) this gives us

$$\sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \left(\frac{d}{dt} \delta q_i - \dot{q}_i \frac{d}{dt} \delta t \right) \right] + \frac{\partial L}{\partial t} \delta t + L \frac{d}{dt} \delta t = \frac{d}{dt} \delta G \quad (2)$$

where F has been expanded in the infinitesimals as δG , which can be written using functions $G_q(q, t)$ and $G_t(q, t)$ as

$$\frac{d}{dt} \delta G = \frac{dG_q}{dt} \delta q + \frac{dG_t}{dt} \delta t$$

We should also include the possibility that F is not due to coordinate transformation, e.g. from gauge transformation

Given an expression for L Eq.(2) tells us what all the (infinitesimal) symmetries of the Lagrangian are in terms of δq_i and δt before the e.o.m. are applied

These are called **off-shell** symmetries and are symmetries of the action

We are interested in the conserved quantities under these symmetries for particles that obey the e.o.m.

Using the Lagrange equations we can write the total time derivative of L as

$$\begin{aligned}\frac{dL}{dt} &= \sum_i \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right) + \frac{\partial L}{\partial t} = \sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right] + \frac{\partial L}{\partial t} \\ &= \sum_i \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) + \frac{\partial L}{\partial t}\end{aligned} \quad (3)$$

With this we can rewrite (2) as

$$\sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \left(\frac{d}{dt} \delta q_i - \dot{q}_i \frac{d}{dt} \delta t \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \delta t \right] + \frac{dL}{dt} \delta t + L \frac{d}{dt} \delta t = \frac{d}{dt} \delta G$$

or

$$\frac{d}{dt} \left[\sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i - \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right) \delta t - \delta G \right] = 0$$

in the book only
the first term is
shown

showing that the **Noether charge**

$$\delta Q \equiv \sum_i \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{p_i} \delta q_i - \underbrace{\left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right)}_{=H} \delta t - \delta G$$

is conserved.

Example

we typically
assume $\delta G = 0$

Assume Lagrangian is invariant under transformation with $\delta t = \delta \epsilon$ and $\delta q_i = 0$, where $\delta \epsilon$ is a constant

$$\delta Q = - \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right) \delta \epsilon = -H \delta \epsilon$$

Here the Hamiltonian H is conserved because ϵ is a constant that can be removed

$$\frac{d}{dt} \delta Q = 0 \Rightarrow \frac{d}{dt} (-H \delta \epsilon) = 0 \Rightarrow \frac{dH}{dt} = 0$$

Example

Assume Lagrangian is invariant under $\delta t = 0$ and $\delta q_i = \delta \epsilon_i$ where $\delta \epsilon_i$ is an arbitrary constant. Then

$$\delta Q = \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta \epsilon_i$$

Since $\delta \epsilon_i$ is an arbitrary constant, the generalised momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

is conserved.

Example: particle in central potential

Lagrangian for particle in central potential

$$K = \frac{1}{2} m \dot{\mathbf{r}}^2, \quad V = V(r) \quad \Rightarrow \quad L = \frac{1}{2} m \dot{\mathbf{r}}^2 - V(r)$$

In spherical coordinates

see Problem
Set 2

$$L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2) - V(r)$$

Obviously symmetric under all rotations, but only one cyclic coordinate ϕ .
Conserved generalised momentum

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2 \theta \dot{\phi}, \quad (z\text{-component of angular momentum})$$

Should not all angular momenta be conserved?

Look at transformation that is infinitesimal rotation around axis $\delta \boldsymbol{\epsilon}$:

$$\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r} + \delta \mathbf{r} = \mathbf{r} + \delta \boldsymbol{\epsilon} \times \mathbf{r}$$

The Cartesian coordinates transform as

$$\begin{aligned} \mathbf{r}^2 &\rightarrow \mathbf{r}'^2 = (\mathbf{r} + \delta \boldsymbol{\epsilon} \times \mathbf{r})^2 \simeq \mathbf{r}^2 + 2\mathbf{r} \cdot \delta \boldsymbol{\epsilon} \times \mathbf{r} = \mathbf{r}^2 \\ \dot{\mathbf{r}}^2 &\rightarrow \dot{\mathbf{r}}'^2 = (\dot{\mathbf{r}} + \delta \boldsymbol{\epsilon} \times \dot{\mathbf{r}})^2 \simeq \dot{\mathbf{r}}^2 + 2\dot{\mathbf{r}} \cdot \delta \boldsymbol{\epsilon} \times \dot{\mathbf{r}} = \dot{\mathbf{r}}^2 \end{aligned}$$

so the Lagrangian, $L[\mathbf{r}', \dot{\mathbf{r}}'] = L[\mathbf{r}, \dot{\mathbf{r}}]$, and thus e.o.m. are invariant under transformation

Noether's theorem gives

Rottmann p. 62

$$\delta Q = \sum_{i=1}^3 \frac{\partial L}{\partial \dot{x}_i} \delta x_i = \sum_{i=1}^3 m x_i \delta x_i = m \mathbf{r} \cdot \delta \mathbf{r} = m \mathbf{r} \cdot (\delta \boldsymbol{\epsilon} \times \mathbf{r}) = m(\mathbf{r} \times \dot{\mathbf{r}}) \cdot \delta \boldsymbol{\epsilon}$$

Since $\delta \boldsymbol{\epsilon}$ is arbitrary, angular momentum $\boldsymbol{\ell} = \mathbf{r} \times m \dot{\mathbf{r}}$ is conserved

More general symmetries

Noether's theorem applies beyond coordinate symmetries to internal symmetries

In Lecture 5 we showed that under gauge transformations of the potential the Lagrangian transforms as

$$L \rightarrow L' = L + q \frac{d\chi}{dt}$$

where $\chi(\mathbf{r}, t)$ is the gauge transformation parameter.

This means that the Lagrangian is symmetric in the sense of (1) with $F(\mathbf{r}, t) = q\chi(\mathbf{r}, t)$

While this is not a coordinate symmetry Noether's theorem still applies when the transformation is parametrised by some small gauge parameter $\delta G = q\delta\chi$ giving

$$\delta Q = q\delta\chi$$

However, drawing further conclusions from this is not possible until we have Noether's theorem for a field theory describing the effect of gauge transformations both on the e.m. fields and the fermion charges they couple to.

Proof of Noether's theorem II (not examinable)

This proof treats Noether's theorem in a more mathematical way

Assume that we have the following coordinate transformations parameterised by a parameter s

$$t \rightarrow t' = t'(q, t, s), \quad q_i \rightarrow q'_i = q'_i(q, t, s),$$

such that for $s = 0$ the transformation reduces to the identity, $t'(q, t, 0) = t$ and $q'_i(q, t, 0) = q_i$.

The function F in (1) can also depend on s which specifies transformation, this includes also transformations that are not coordinate transformations (e.g. gauge)

We now define

$$\theta(q, t) \equiv \left. \frac{\partial t'(q, t, s)}{\partial s} \right|_{s=0}, \quad \eta_i(q, t) \equiv \left. \frac{\partial q'_i(q, t, s)}{\partial s} \right|_{s=0}$$

Starting from the symmetry requirement (1) we differentiate both sides with respect to s and evaluate at $s = 0$. (Finding what the symmetry requirement is near identity)

first order
coefficients in
Taylor
expansion of
transformations
around $s = 0$
(assumes
convergence)

We get for the right hand side

$$\left. \frac{\partial}{\partial s} \frac{dF(q, t, s)}{dt} \right|_{s=0} = \frac{d}{dt} \left. \frac{\partial F(q, t, s)}{\partial s} \right|_{s=0} \quad (4)$$

Schwarz'
theorem,
continuous
second partial
derivatives

This uses that partial differentiation commutes if the function F is well behaved
For the left hand side we get

$$\begin{aligned} \left. \frac{\partial}{\partial s} \left(L[q', \dot{q}', t'] \frac{dt'}{dt} \right) \right|_{s=0} &= \left. \frac{\partial L[q', \dot{q}', t']}{\partial s} \frac{dt'}{dt} \right|_{s=0} + L[q', \dot{q}', t'] \left. \frac{\partial}{\partial s} \frac{dt'}{dt} \right|_{s=0} \\ &= \left. \frac{\partial L[q', \dot{q}', t']}{\partial s} \right|_{s=0} + L[q, \dot{q}, t] \left(\left. \frac{\partial}{\partial s} \frac{dt'}{dt} \right)_{s=0} \right) \end{aligned} \quad (5)$$

Here (again Schwarz)

$$\left. \frac{\partial}{\partial s} \frac{dt'}{dt} \right|_{s=0} = \left. \frac{d}{dt} \frac{\partial t'}{\partial s} \right|_{s=0} = \frac{d\theta}{dt}$$

and

requires L to
have a
convergent
Taylor series
around $s = 0$

$$\begin{aligned} \left. \frac{\partial L[q', \dot{q}', t']}{\partial s} \right|_{s=0} &= \sum_i \left(\frac{\partial L[q', \dot{q}', t']}{\partial q'_i} \frac{\partial q'_i}{\partial s} + \frac{\partial L[q', \dot{q}', t']}{\partial \dot{q}'_i} \frac{\partial \dot{q}'_i}{\partial s} \right)_{s=0} + \left. \frac{\partial L[q', \dot{q}', t']}{\partial t'} \frac{\partial t'}{\partial s} \right|_{s=0} \\ &= \sum_i \left[\frac{\partial L[q, \dot{q}, t]}{\partial q_i} \eta_i + \frac{\partial L[q, \dot{q}, t]}{\partial \dot{q}_i} \left(\frac{d\eta_i}{dt} - \dot{q}_i \frac{d\theta}{dt} \right) \right] + \frac{\partial L[q, \dot{q}, t]}{\partial t} \theta \end{aligned}$$

where we have used that

$$\left(\frac{\partial \dot{q}'_i}{\partial s} \right)_{s=0} = \frac{d\eta_i}{dt} - \dot{q}_i \frac{d\theta}{dt}$$

Gathering (5) and (4), and simplifying notation by writing $L[q, \dot{q}, t] = L$, we get

$$\sum_i \left[\frac{\partial L}{\partial q_i} \eta_i + \frac{\partial L}{\partial \dot{q}_i} \left(\frac{d\eta_i}{dt} - \dot{q}_i \frac{d\theta}{dt} \right) \right] + \frac{\partial L}{\partial t} \theta + L \frac{d\theta}{dt} = \frac{dG}{dt} \quad (6)$$

where we have defined

$$G(q, t, s) \equiv \left. \frac{\partial F(q, t, s)}{\partial s} \right|_{s=0}$$

Once L has been specified this equation is a (difficult!) differential equation for θ and η_i . Solving it will give all the **off-shell** symmetries of the Lagrangian.

We are now interested in the conserved quantities under these symmetries.

We use (3) rewriting (6) as the **on-shell** symmetry condition

$$\sum_i \left[\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \eta_i + \frac{\partial L}{\partial \dot{q}_i} \left(\frac{d\eta_i}{dt} - \dot{q}_i \frac{d\theta}{dt} \right) - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) \theta \right] + \frac{dL}{dt} \theta + L \frac{d\theta}{dt} = \frac{dG}{dt}$$

or

$$\frac{d}{dt} \left[\sum_i \frac{\partial L}{\partial \dot{q}_i} \eta_i - \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right) \theta - G \right] = 0$$

showing that the **Noether charge**

$$Q = \sum_i \frac{\partial L}{\partial \dot{q}_i} \eta_i - \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right) \theta - G$$

or

$$Q = \sum_i p_i \eta_i - H \theta - G$$

is conserved.