

FYS3120 Classical Mechanics and
Electrodynamics

Solutions for Problem set 1

January 25, 2025

Problem 1 Figure 1 shows four different mechanical systems, in **a)** a pendulum attached to a block which in turn is attached to a spring, **b)** a pendulum which is attached to a vertical ring which rotates with a fixed frequency ω , **c)** a straight rod which can tilt without sliding on the top of cylinder, while the cylinder can roll on a horizontal plane (no slipping), and **d)** a spinning top which moves on a horizontal floor.

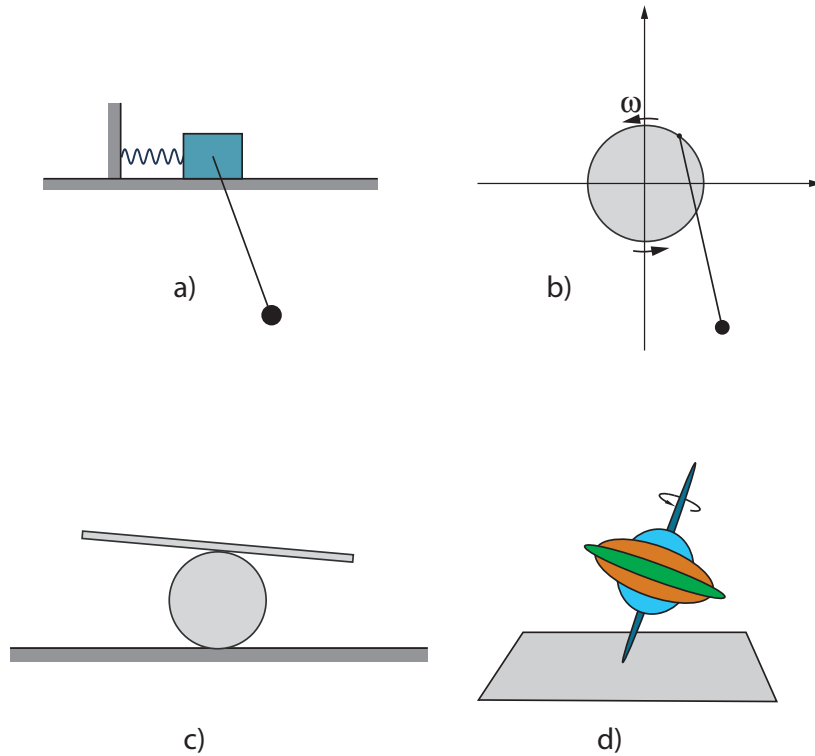


Figure 1: Four mechanical systems.

In all cases specify the number of degrees of freedom and choose an appropriate set of generalised coordinates.

Solution: For each system we will begin by counting the number of degrees of freedom using physical considerations, which suffices as an answer to the problem. Following this we will give more rigorous arguments in terms of coordinates and constraints. Finally, we suggest appropriate generalised coordinates.

Grading: Maximum score is 1 point per subproblem, *i.e.* maximum 4 points in total for a)-d):

- Give 0.5 points for correct specification of the number of degrees of freedom.
- Give 0.5 points per subproblem for a reasonable choice of generalised coordinates.

- If a subproblem has correct specification of degrees of freedom and generalised coordinates, but the argument for the number of degrees of freedom is not given (or is incorrect): give 0.5 points only (do not do this for more than two subproblems, even if an argument is missing for all four subproblems).

- a)** This system has two degrees of freedom. The block can move horizontally along the surface, while the pendulum can rotate around the centre of the block with a particular angle.

Since all the movement is in a plane, we can look at this problem in a two dimensional space. Initially, the system consists of the block, which is an extended object in two dimensions, and a point particle, the pendulum. This requires two positions with in all four coordinates, and an angle ϕ to describe the rotation of the block. We then demand that the block lies flat against the surface, which imposes a constraint on its y coordinate ($y = 0$), and its rotation ($\phi = 0$). The pendulum point is constrained to lie at a fixed distance from the block, giving one more constraint. In total this gives five coordinates and three constraints, leaving two degrees of freedom.

As generalised coordinates we can choose the horizontal position x of the block and the angle θ which the pendulum makes with the vertical. There are of course other possible choices, in particular for defining the angle of the pendulum.

- b)** This system has a single degree of freedom, given by the motion of the pendulum in terms of its angle in the plane. Note that the rotation of the disk does not give a degree of freedom, as the angular speed ω is fixed. Another way to look at this is to also consider the angle of the disk, but this has a trivial solution to its equation of motion, namely that the angle ϕ is given as $\phi(t) = \phi_0 + \omega t$, where ϕ_0 is the initial position.

More formally, we have an extended object (the disk) and a point (the pendulum) in two dimensions, giving five degrees of freedom as in the previous system. The position of the disk is fixed at the origin, giving two constraints. The angle ϕ of the disk is fixed by the constraint $\phi - \phi_0 - \omega t = 0$, which expresses the fact that it rotates with a fixed speed ω . Finally, the distance from the pendulum to the rotating attachment point is fixed, giving one more constraint. The result is $5 - 4 = 1$ degree of freedom.

A good generalised coordinate for this system is the angle θ which the pendulum makes with the vertical.

- c)** In this system we can see two obvious degrees of freedom: The cylinder is allowed to roll back and forth with an angular coordinate, and the rod is allowed to tilt atop of it giving another angular coordinate.

Here we have two extended objects in two dimensions, both of which require two positions and a rotation angle to describe, for a total of six coordinates.

The cylinder is constrained to lie on the surface, giving a constraint $y = 0$. Additionally, it is required to roll without slipping, giving a constraint between the distance rolled for the centre and the angle, of the type $x = R\phi$, with R being the radius of the cylinder and ϕ the angle. Setting up explicit constraints for the rod is more challenging since we have not specified its dimensions and the position of the point of contact with the cylinder. However, if we know the angle θ which the rod makes with the horizontal, we can uniquely determine both its horizontal and vertical position relative to the cylinder. This should therefore give rise to two constraints, one for the horizontal and one for the vertical position. In total we have $6 - 4 = 2$ degrees of freedom.

As generalised coordinates one can choose the horizontal position x of the cylinder and the angle θ which the rod makes with the horizontal.

- d) This is a three-dimensional rigid body, which can move across a horizontal plane and rotate about any of its axes. The horizontal motion gives two degrees of freedom, and there are three independent angles that specify the position of the top, the angle of rotation of the top around its own central axis, the angle to the vertical and the angle of the central axis in the horizontal plane, giving five degrees of freedom in total.

We get almost exactly the same argument from counting constraints: A three-dimensional rigid body is described by three positional and three rotational coordinates, and there is the single constraint that the bottom point must lie on the surface.

As generalised coordinates, one can choose the position (x, y) of the bottom point, and three angles which describe the direction of the spinning top in space. For these angles we can choose the angle θ which the central axis of the spinning top makes with the z axis, the polar angle ϕ of the central axis when projected down in the (x, y) -plane, and the angle χ of the spinning top around its central axis.

Problem 2 A particle with mass m moves in three-dimensional space under the influence of a constraint. The constraint is expressed by the equation

$$e^{-(x^2+y^2)} + z = 0, \quad (1)$$

for the Cartesian coordinates (x, y, z) of the particle.

- a) Explain why the number of degrees of freedom of the particle is two. Use x and y as generalised coordinates and find the expression for the position vector $\mathbf{r}$ of the particle in terms of x and y .

Solution: A particle in three-dimensional space has three physical coordinates. The single constraint yields $3 - 1 = 2$ degrees of freedom. Choosing

x and y as generalised coordinates, we need to determine $\mathbf{r}(x, y)$. The x and y coordinates of $\mathbf{r}$ are just the generalised coordinates themselves, so it suffices to determine $z(x, y)$. This we find from the constraint equation as $z(x, y) = -e^{-(x^2+y^2)}$, so

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} - e^{-(x^2+y^2)}\mathbf{k}.$$

Grading: Maximum score is 1 point for this subproblem:

- Give 0.5 points for a correct argument for the number of degrees of freedom.
 - Give 0.5 points for the correct position vector as function of x and y .
- b) A virtual displacement is a change in the position of the particle $\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r} + \delta\mathbf{r}$ which is caused by an infinitesimal change in the generalised coordinates, $x \rightarrow x' = x + \delta x$ and $y \rightarrow y' = y + \delta y$. Find $\delta\mathbf{r}$ expressed in terms of δx and δy .

Solution: Given a virtual displacement $x \rightarrow x + \delta x$, $y \rightarrow y + \delta y$, the position vector $\mathbf{r}(x, y)$ will change to $\mathbf{r}(x + \delta x, y + \delta y) = \mathbf{r}(x, y) + \delta\mathbf{r}$. Expanding $\mathbf{r}(x + \delta x, y + \delta y)$ as a Taylor series around (x, y) and using that the displacements are infinitesimal, we obtain

$$\delta\mathbf{r} = \mathbf{r}(x + \delta x, y + \delta y) - \mathbf{r}(x, y) = \frac{\partial\mathbf{r}}{\partial x}\delta x + \frac{\partial\mathbf{r}}{\partial y}\delta y.$$

Inserting for $\mathbf{r}$,

$$\begin{aligned}\delta\mathbf{r} &= \delta x\mathbf{i} + \delta y\mathbf{j} + \left(\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right)\mathbf{k} \\ &= \delta x\mathbf{i} + \delta y\mathbf{j} + (2x\delta x + 2y\delta y)e^{-(x^2+y^2)}\mathbf{k}.\end{aligned}\tag{2}$$

Grading: Maximum score is 1 point for this subproblem:

- Give 0.5 points for using that $\delta\mathbf{r} = \frac{\partial\mathbf{r}}{\partial x}\delta x + \frac{\partial\mathbf{r}}{\partial y}\delta y$.
 - Give 0.5 points for correctly calculating $\delta\mathbf{r}$.
- c) The constraint can be interpreted as a restriction for the particle to move on a two-dimensional surface in three-dimensional space. Any virtual displacement $\delta\mathbf{r}$ is a tangent vector to the surface while the constraint force $\mathbf{f}$ which acts on the particle is perpendicular to the surface. Use this to determine $\mathbf{f}$ (up to a normalisation factor) as a function of x and y .

Solution: Since $\delta\mathbf{r}$ is tangent to the surface, and $\mathbf{f} = (f_x, f_y, f_z)$ must be normal to the surface, in particular $\mathbf{f}$ must be normal to $\delta\mathbf{r}$. This gives the equation

$$\mathbf{f} \cdot \delta\mathbf{r} = f_x\delta x + f_y\delta y + f_z(2x\delta x + 2y\delta y)e^{-(x^2+y^2)} = 0.$$

Gathering terms proportional to each independent displacement,

$$\left(f_x + 2xf_z e^{-(x^2+y^2)}\right)\delta x + \left(f_y + 2yf_z e^{-(x^2+y^2)}\right)\delta y = 0.$$

Different displacements δx and δy gives rise to different tangent vectors $\delta \mathbf{r}$. Since $\mathbf{f}$ must be normal to any such tangent vector, the above equation must be valid for all δx and δy , so the two brackets must vanish independently, giving

$$f_x = -2xf_z e^{-(x^2+y^2)}, \quad f_y = -2yf_z e^{-(x^2+y^2)}.$$

This determines the constraint force as

$$\mathbf{f} = f_z(-2xe^{-(x^2+y^2)}, -2ye^{-(x^2+y^2)}, 1),$$

where the number f_z is an undetermined normalisation factor.

Grading: Maximum score is 1 point for this subproblem:

- Give 0.5 points for realizing that $\mathbf{f} \cdot \delta \mathbf{r} = 0$.
 - Give 0.5 points for a correct calculation that leads to a correct expression for $\mathbf{f}$.
- d) Make a drawing of a section through the surface for $y = 0$. Indicate in the drawing the direction of the two vectors $\mathbf{f}$ and $\delta \mathbf{r}$ for a chosen point on the surface.

Solution: Setting $y = 0$ in $z(x, y)$, we see that

$$z(x, 0) = -e^{-x^2}.$$

This is an upside-down Gaussian curve, as plotted below. At a point $(x, z(x))$, the tangent vector $\delta \mathbf{r}$ points along the curve, while the force $\mathbf{f}$ is normal to the curve. These vectors are drawn into the plot as arrows starting from this point.

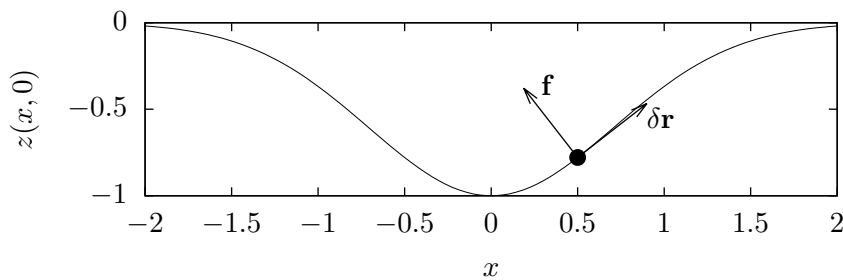


Figure 2: Section through the surface $z(x, y)$ at $y = 0$.

Grading: Maximum score is 1 point for this subproblem:

- Give 0.5 points for a correct surface, or correct cross section of the surface.
- Give 0.5 points for displaying the directions of $\mathbf{f}$ and $\delta\mathbf{r}$.

Problem 3 A pendulum consists of a rigid rod, which we consider as massless, and a pendulum bob of mass m . The point of suspension of the pendulum has horizontal coordinate $x = s$ and vertical coordinate $y = 0$. This set-up is shown in Fig. 3.

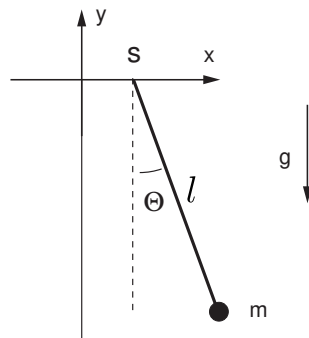


Figure 3: The pendulum problem.

- a) Assume first that the point of suspension is kept fixed, with $s = 0$. Use the angle θ as generalised coordinate, find the Lagrangian of the system and determine the form of Lagrange's equation for the system. Check that it has the standard form of a pendulum equation.

Solution: The pendulum bob has speed $l\dot{\theta}$ and height $-l \cos \theta$, giving the Lagrangian

$$L = K - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl \cos \theta.$$

Lagrange's equation gives

$$ml^2\ddot{\theta} + mgl \sin \theta = 0,$$

or, simplifying,

$$\ddot{\theta} + \frac{g}{l} \sin \theta = 0,$$

which is the form of a pendulum equation.

Grading: Maximum score is 2 points for this subproblem:

- 1 point for the correct Lagrangian: 0.5 for correct kinetic energy, and 0.5 points for correct potential energy (with correct sign).

- 1 point for the correct equation of motion: 0.5 points for correct $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right)$ -part, and 0.5 for correct $\frac{\partial L}{\partial \theta}$ -part.

b) Find and categorise any equilibrium points for the system.

Solution: Equilibrium points are given by the extremum points of the potential in terms of the generalised coordinates:

$$\frac{\partial V}{\partial q} = \frac{\partial V}{\partial \theta} = l \sin \theta = 0.$$

This gives $\theta_0 = 0$ and $\theta_0 = \pi$ as physically separate equilibrium points. The second derivative

$$\frac{\partial^2 V}{\partial \theta^2} = l \cos \theta,$$

means that $\theta_0 = 0$ is a stable equilibrium (minimum) and $\theta_0 = \pi$ is unstable (maximum).

Grading: Maximum score is 1 point for this subproblem:

- 0.5 for finding the correct equilibrium points using the potential.
 - 0.5 for finding their types through the second derivative.
- c)** The point of suspension is now released so it can move freely in the horizontal direction (x -direction). Use s and θ as generalised coordinates for the system and determine the corresponding set of Lagrange's equations.

Solution: The pendulum bob now has position $x = s + l \sin \theta$ and $y = -l \cos \theta$. This gives the kinetic energy

$$\begin{aligned} K &= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \left[\left(\dot{s} + l \dot{\theta} \cos \theta \right)^2 + l^2 \dot{\theta}^2 \sin^2 \theta \right] \\ &= \frac{1}{2} m \left(\dot{s}^2 + 2l\dot{s}\dot{\theta} \cos \theta + l^2 \dot{\theta}^2 \right). \end{aligned} \quad (3)$$

The potential energy is unchanged, giving

$$L = K - V = \frac{1}{2} m \left(\dot{s}^2 + 2l\dot{s}\dot{\theta} \cos \theta + l^2 \dot{\theta}^2 \right) + mgl \cos \theta.$$

The s coordinate is cyclic, which gives the conserved quantity

$$p_s = \frac{\partial L}{\partial \dot{s}} = m\dot{s} + ml\dot{\theta} \cos \theta.$$

This is the equation of motion for s , which is perhaps easier to see if solved for $\dot{s}$:

$$\dot{s} = \frac{p_s}{m} - l\dot{\theta} \cos \theta, \quad (4)$$

where p_s is a constant. This can be solved for $s(t)$ if $\theta(t)$ is known. If we recall from above that $\dot{x} = \dot{s} + l\dot{\theta} \cos \theta$, we can see that in fact the conserved quantity is $p_s = m\dot{x}$. This means that it is the horizontal momentum of the pendulum bob that is conserved.

The equation for θ is harder to deal with. We have that

$$\frac{\partial L}{\partial \dot{\theta}} = ml\dot{s} \cos \theta + ml^2\dot{\theta}, \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = ml\ddot{s} \cos \theta - ml\dot{s}\dot{\theta} \sin \theta + ml^2\ddot{\theta}, \quad (5)$$

and

$$\frac{\partial L}{\partial \theta} = -ml\dot{s}\dot{\theta} \sin \theta - mgl \sin \theta, \quad (6)$$

so Lagrange's equation for the θ coordinate is

$$ml\ddot{s} \cos \theta - ml\dot{s}\dot{\theta} \sin \theta + ml^2\ddot{\theta} + ml\dot{s}\dot{\theta} \sin \theta + mgl \sin \theta = 0.$$

The two terms $ml\dot{s}\dot{\theta} \sin \theta$ cancel each other. We can also cancel a factor ml , leaving

$$l\ddot{\theta} + \ddot{s} \cos \theta + g \sin \theta = 0. \quad (7)$$

Grading: Maximum score is 3 points for this subproblem:

- 0.5 points for the correct x - and y -coordinates in terms of s and θ .
 - 0.5 points for correct Lagrangian.
 - 1 point for the equation of motion for s .
 - 1 point for the equation of motion for θ .
- d) Show that s can be eliminated to give an equation which only depends on θ and its derivatives. Further show that this equation implies that the vertical motion of the pendulum bob is identical to free fall in the gravitational field (in reality restricted by the length l of the rod). *Warning:* Do not attempt to substitute directly into the Lagrangian. Try to think why this will give the wrong answer.

Solution: We need to eliminate $\ddot{s}$ from Eq. (7). To do this, recall that $p_s = m\dot{s} + ml\dot{\theta} \cos \theta$ is constant in time. Cancelling a factor m and differentiating, we find that

$$\ddot{s} = -l \frac{d}{dt} (\dot{\theta} \cos \theta) = l\dot{\theta}^2 \sin \theta - l\ddot{\theta} \cos \theta.$$

This identity can be inserted directly into Eq. (7), giving

$$\begin{aligned} (l\dot{\theta}^2 \sin \theta - l\ddot{\theta} \cos \theta) \cos \theta + l\ddot{\theta} + g \sin \theta &= 0, \\ l\ddot{\theta} (1 - \cos^2 \theta) + l\dot{\theta}^2 \sin \theta \cos \theta + g \sin \theta &= 0, \\ l\ddot{\theta} \sin^2 \theta + l\dot{\theta}^2 \cos \theta \sin \theta + g \sin \theta &= 0. \end{aligned} \quad (8)$$

The reason we can not substitute $\dot{s}$ directly into the Lagrangian is that Lagrange's equation has partial derivatives, implying that the other variable(s) are kept constant in forming the equation of motion. Inserting the relationship for $\dot{s}$ in (4) implies that $\dot{s}$ changes when we change θ (or $\dot{\theta}$) in Lagrange's equation for θ , violating the assumptions behind the equation.

The statement that the vertical motion of the pendulum bob is identical to free fall is the same as saying that the y position has constant acceleration $\ddot{y} = -g$. Recall that $y = -l \cos \theta$. Differentiating twice,

$$\ddot{y} = l \frac{d}{dt} (\dot{\theta} \sin \theta) = l \dot{\theta}^2 \cos \theta + l \ddot{\theta} \sin \theta.$$

We can now see that the first two terms of the above equation is exactly $\ddot{y}$ multiplied by $\sin \theta$:

$$\ddot{y} \sin \theta = -g \sin \theta.$$

If $\sin \theta \neq 0$ we can cancel the factor $\sin \theta$ from both sides, proving the statement:

$$\ddot{y} = -g.$$

It is worthwhile for the reader to think about why this argument only works for $\sin \theta \neq 0$.

Grading: Maximum score is 3 points for this subproblem:

- 1 points for finding the correct expression for $\ddot{s}$ and inserting in Lagrange's equation for θ .
- 0.5 points for correctly simplifying the algebra to find a differential equation that only involves θ , similar to equation (8).
- 0.5 points for realizing that the equation for free fall should include the y -coordinate.
- 0.5 points for finding $\ddot{y}$ in terms of θ .
- 0.5 points for substituting $\ddot{y}$ in the equation of motion for θ and showing how this equation looks like free fall in the gravitational field.