

FYS3120 Classical Mechanics and
Electrodynamics

Solutions for Problem set 2

February 1, 2025

Problem 1 Reduced mass

Let us look at the generic two-body problem with two objects of mass m_1 and m_2 . Assume that the potential (force) between the two bodies is only dependent on the distance between the two objects – a so-called **central potential** – as would be the case for both gravitational and electrostatic potentials (forces). There are no constraints.

- a) Write down the Lagrangian L in terms of the coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ of the two objects.

Solution: Since the potential is dependent only on the distance $|\mathbf{r}_1 - \mathbf{r}_2|$, we must have

$$L = K - V = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 - V(|\mathbf{r}_1 - \mathbf{r}_2|). \quad (1)$$

Grading: Maximum score is 1 point for this subproblem:

- 0.5 points for the kinetic terms and 0.5 for the potential.

- b) Make the change of variables to the (generalised) coordinates $(\mathbf{r}, \mathbf{R})$, where

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad (2)$$

$$\mathbf{R} = \frac{m_1}{m_1 + m_2}\mathbf{r}_1 + \frac{m_2}{m_1 + m_2}\mathbf{r}_2, \quad (3)$$

$\mathbf{R}$ being the **centre-of-mass** of the two bodies, and show that the resulting Lagrangian is

$$L = \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2 - V(r), \quad (4)$$

where $M = m_1 + m_2$ is the **total mass**, and

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad (5)$$

is the **reduced mass**.

Solution: The change of variables can be solved for $\mathbf{r}_1$ and $\mathbf{r}_2$:

$$\mathbf{R} + \frac{m_2}{m_1 + m_2}\mathbf{r} = \frac{m_1}{m_1 + m_2}\mathbf{r}_1 + \frac{m_2}{m_1 + m_2}\mathbf{r}_1 = \mathbf{r}_1, \quad (6)$$

$$\mathbf{R} - \frac{m_1}{m_1 + m_2}\mathbf{r} = \frac{m_2}{m_1 + m_2}\mathbf{r}_2 + \frac{m_1}{m_1 + m_2}\mathbf{r}_2 = \mathbf{r}_2. \quad (7)$$

Taking the derivative with respect to time and inserting in the original Lagrangian (1) this gives

$$L = \frac{1}{2}m_1 \left(\dot{\mathbf{R}} + \frac{m_2}{m_1 + m_2}\dot{\mathbf{r}} \right)^2 + \frac{1}{2}m_2 \left(\dot{\mathbf{R}} - \frac{m_1}{m_1 + m_2}\dot{\mathbf{r}} \right)^2 - V(r)$$

$$\begin{aligned}
&= \frac{1}{2}m_1 \left(\dot{\mathbf{R}}^2 + \frac{2m_2}{m_1 + m_2} \dot{\mathbf{R}}\dot{\mathbf{r}} + \left(\frac{m_2}{m_1 + m_2} \right)^2 \dot{\mathbf{r}}^2 \right) \\
&\quad + \frac{1}{2}m_2 \left(\dot{\mathbf{R}}^2 - \frac{2m_1}{m_1 + m_2} \dot{\mathbf{R}}\dot{\mathbf{r}} + \left(\frac{m_1}{m_1 + m_2} \right)^2 \dot{\mathbf{r}}^2 \right) - V(r) \\
&= \frac{1}{2}(m_1 + m_2)\dot{\mathbf{R}}^2 + \frac{1}{2} \frac{m_1 m_2^2 + m_1^2 m_2}{(m_1 + m_2)^2} \dot{\mathbf{r}}^2 - V(r) \\
&= \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \dot{\mathbf{r}}^2 - V(r). \tag{8}
\end{aligned}$$

Grading: Maximum score is 2 points for this subproblem:

- 1 point for inverting the coordinates.
 - 1 point for correct insertion and showing the resulting form of the Lagrangian.
 - Up to 2 points, if it is shown instead by starting from equation (4) and inserting, or in an other correct way.
- c) Find Lagrange's equations in the new coordinates and show that the solution for the path of $\mathbf{R}$ is

$$\mathbf{R}(t) = \mathbf{R}_0 + \mathbf{v}_0 t, \tag{9}$$

where $\mathbf{R}_0$ is some initial position at $t = 0$ and $\mathbf{v}_0$ is the initial velocity.

Solution:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{R}_i} - \frac{\partial L}{\partial R_i} = (m_1 + m_2) \ddot{R}_i = 0, \tag{10}$$

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{r}_i} - \frac{\partial L}{\partial r_i} = \mu \ddot{r}_i - \frac{\partial V}{\partial r_i} = 0, \tag{11}$$

for $i = 1, 2, 3$. Since $\ddot{R}_i = 0$ these coordinates must have some constant velocity, meaning $\dot{\mathbf{R}} = \mathbf{v}_0$. After integrating again, the general solution for $\mathbf{R}$ must be $\mathbf{R}(t) = \mathbf{R}_0 + \mathbf{v}_0 t$, where $\mathbf{R}_0$ is some initial position at $t = 0$ and $\mathbf{v}_0$ is the initial velocity.

Grading: Maximum score is 3 points for this subproblem:

- 1 point for Lagrange's equations for R_i .
 - 1 point for Lagrange's equations for r_i .
 - 1 point for the solution for $\mathbf{R}(t)$.
- d) What is the physical interpretation of the equations of motion? Argue why it is sufficient to study only the Lagrangian

$$L = \frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(r), \tag{12}$$

in the two-body problem.

Solution: These new coordinates consist of the coordinates of the centre-of-mass of the two objects, $\mathbf{R}$, and the distance between them, $\mathbf{r}$. The change of variables gives an e.o.m. describing the motion of the centre-of-mass as a free body with total mass $M = m_1 + m_2$, meaning that the centre-of-mass has constant linear velocity and conserved (mechanical) momentum because the e.o.m. (and the Lagrangian) is invariant under constant translations of $\mathbf{R}$.

At the same time, the relative motion of the two objects is described as a body with the reduced mass μ , but with the same potential. This proves that the two-body problem may always be reduced to the independent free motion of the centre-of-mass of the two bodies, and their relative motion.

To find the resulting motion it is then sufficient to look at the part of the Lagrangian with the one-body relative coordinates $\mathbf{r}$, since if the motion of $\mathbf{r}(t)$ is found, the motion of the centre-of-mass system could just be added by using Eqs. (7) and (8) to give a solution

$$\mathbf{r}_1(t) = \frac{m_2}{m_1 + m_2} \mathbf{r}(t) + \mathbf{R}(t), \quad (13)$$

$$\mathbf{r}_2(t) = -\frac{m_1}{m_1 + m_2} \mathbf{r}(t) + \mathbf{R}(t). \quad (14)$$

Grading: Maximum score is 3 points for this subproblem:

- 1 point for the interpretation of the motion of the centre-of-mass coordinates.
- 1 point for the interpretation of the motion in the relative coordinate, including the reduced mass concept.
- 1 point for a good explanation of why it is sufficient to find the motion of the relative coordinate.

e) Express the Lagrangian in Eq. (13) in terms of spherical coordinates, where $\mathbf{r} = (x, y, z)$ with

$$x = r \cos \phi \sin \theta,$$

$$y = r \sin \phi \sin \theta,$$

$$z = r \cos \theta.$$

Use this to show that the angular momentum in the z -direction, ℓ_z , is conserved.

Solution: In spherical coordinates the velocities are,

$$\dot{x} = \dot{r} \cos \phi \sin \theta - r \sin \phi \sin \theta \dot{\phi} + r \cos \phi \cos \theta \dot{\theta},$$

$$\dot{y} = \dot{r} \sin \phi \sin \theta + r \cos \phi \sin \theta \dot{\phi} + r \sin \phi \cos \theta \dot{\theta},$$

$$\dot{z} = \dot{r} \cos \theta - r \sin \theta \dot{\theta},$$

Squaring and doing some tidying using $\sin^2 \phi + \cos^2 \phi = 1$,

$$\begin{aligned}
\dot{x}^2 &= \dot{r}^2 \cos^2 \phi \sin^2 \theta - 2\dot{r}r\dot{\phi} \cos \phi \sin \phi \sin^2 \theta + 2\dot{r}r\dot{\theta} \cos^2 \phi \sin \theta \cos \theta \\
&\quad + r^2 \dot{\phi}^2 \sin^2 \phi \sin^2 \theta - 2r^2 \dot{\theta} \dot{\phi} \sin \phi \cos \phi \sin \theta \cos \theta + r^2 \dot{\theta}^2 \cos^2 \phi \cos^2 \theta \\
\dot{y}^2 &= \dot{r}^2 \sin^2 \phi \sin^2 \theta + 2\dot{r}r\dot{\phi} \sin \phi \cos \phi \sin^2 \theta + 2\dot{r}r\dot{\theta} \sin^2 \phi \sin \theta \cos \theta \\
&\quad + r^2 \dot{\phi}^2 \cos^2 \phi \sin^2 \theta + 2r^2 \dot{\theta} \dot{\phi} \cos \phi \sin \phi \sin \theta \cos \theta + r^2 \dot{\theta}^2 \sin^2 \phi \cos^2 \theta \\
\dot{z}^2 &= \dot{r}^2 \cos^2 \theta - 2\dot{r}r\dot{\theta} \cos \theta \sin \theta + r^2 \dot{\theta}^2 \sin^2 \theta \\
\dot{x}^2 + \dot{y}^2 &= \dot{r}^2 \sin^2 \theta + 2\dot{r}r\dot{\theta} \sin \theta \cos \theta + r^2 \dot{\phi}^2 \sin^2 \theta + r^2 \dot{\theta}^2 \cos^2 \theta
\end{aligned}$$

gives

$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta, \quad (15)$$

and the Lagrangian can then be expressed as

$$L = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \dot{\phi}^2 \sin^2 \theta) - V(r). \quad (16)$$

We immediately see that ϕ is a cyclic coordinate. This means that the generalised momentum

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \mu r^2 \sin^2 \theta \dot{\phi}, \quad (17)$$

is conserved. The angular momentum for a mass μ in the z -direction (rotation around z -axis) is

$$\begin{aligned}
\ell_z &\equiv \mu(x\dot{y} - y\dot{x}) \\
&= \mu(r\dot{r} \sin \phi \cos \phi \sin^2 \theta + r^2 \dot{\phi} \cos^2 \phi \sin^2 \theta + r^2 \dot{\theta} \sin \phi \cos \phi \sin \theta \cos \theta \\
&\quad - r\dot{r} \sin \phi \cos \phi \sin^2 \theta + r^2 \dot{\phi} \sin^2 \phi \sin^2 \theta - r^2 \dot{\theta} \sin \phi \cos \phi \sin \theta \cos \theta) \\
&= \mu r^2 \sin^2 \theta \dot{\phi}
\end{aligned}$$

so the conserved quantity is angular momentum in the z -direction.

Grading: Maximum score is 2 points for this subproblem:

- 1 point for finding the Lagrangian.
- 1 for identifying the cyclic coordinate and showing that the conserved quantity is angular momentum.

- e) In fact, since the choice of the z -axis in this problem is arbitrary, all components of angular momentum must be conserved.¹ This means in turn that the motion in the coordinate $\mathbf{r}$ must be in the plane perpendicular to the angular momentum – motion out of this plane would generate an angular momentum component in a different direction.

¹We shall show this more formally for central potentials later in the course.

Choose the z -axis to point in the direction of angular momentum. Use this to show that the equation of the motion of the whole system can be reduced to the single differential equation

$$\mu\ddot{r} + \frac{\partial V_{\text{eff}}}{\partial r} = 0, \quad (18)$$

with the **effective potential** V_{eff} given by

$$V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2\mu r^2}, \quad (19)$$

where ℓ is the angular momentum.

Solution: To find the equation of motion for r we use

$$\frac{\partial L}{\partial \dot{r}} = \mu\dot{r}, \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{r}} = \mu\ddot{r}, \quad \frac{\partial L}{\partial r} = r\dot{\theta}^2 + r\dot{\phi}^2 \sin^2 \theta - \frac{\partial V(r)}{\partial r}, \quad (20)$$

which from Lagrange's equation gives

$$\mu\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2 \theta + \frac{\partial V(r)}{\partial r} = 0. \quad (21)$$

With the angular momentum in the z -direction the motion is in the (x, y) -plane, and thus $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$. This reduces the e.o.m. to

$$\mu\ddot{r} - r\dot{\phi}^2 + \frac{\partial V(r)}{\partial r} = 0. \quad (22)$$

Inserting for the conserved angular momentum $\ell = \mu r^2 \sin^2 \theta \dot{\phi} = \mu r^2 \dot{\phi}$ to remove $\dot{\phi}$ we get

$$\mu\ddot{r} - \frac{\ell^2}{\mu^2 r^3} + \frac{\partial V(r)}{\partial r} = \mu\ddot{r} + \frac{\partial V_{\text{eff}}(r)}{\partial r} = 0. \quad (23)$$

With a solution for $r(t)$ the solution for ϕ can now be found from $\ell = \mu r^2 \dot{\phi}$.

Grading: Maximum score is 1 point for this subproblem:

- 0.5 point for the correct Lagrange equation
- 0.5 point for reducing the Lagrange equation to the effective form.

Problem 2 Two bodies with the same mass, m , are connected with a massless rope through a small hole in a smooth horizontal plane. This is illustrated in Fig. 1. One body is moving on the plane, the other one is hanging at the end of the rope and can move vertically. At all instances the rope is tight. The acceleration due to gravity is g .

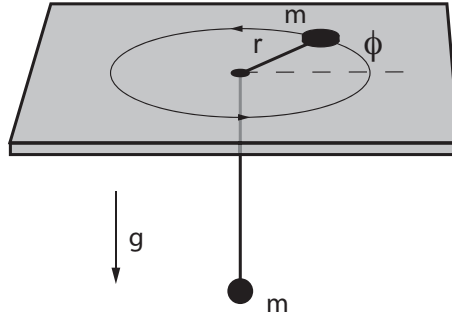


Figure 1: Two bodies connected by a rope through a small hole.

- a) Find Lagrange's equations of motion in polar coordinates (r, ϕ) .

Answer: The position of the upper body is given by

$$x_u = r \cos \phi, \quad y_u = r \sin \phi, \quad z_u = 0.$$

The lower body only moves on the z axis, and has position $z_l = -(l - r) = r - l$. The kinetic energy is then

$$\begin{aligned} K &= \frac{1}{2}m (\dot{x}_u^2 + \dot{y}_u^2 + \dot{z}_l^2) \\ &= \frac{1}{2}m \left[(\dot{r} \cos \phi - r \dot{\phi} \sin \phi)^2 + (\dot{r} \sin \phi + r \dot{\phi} \cos \phi)^2 + \dot{r}^2 \right] \\ &= \frac{1}{2}m (2\dot{r}^2 + r^2 \dot{\phi}^2). \end{aligned} \quad (24)$$

Subtracting the potential energy $V = mgz_l = mg(r - l)$, we find the Lagrangian

$$L = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - mgr + mgl.$$

The ϕ coordinate is cyclic, so $p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}$ is conserved and gives the equation of motion for ϕ . Physically, this quantity is the angular momentum of the upper body around the center point. For the r equation we get

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 2m\ddot{r} - mr\dot{\phi}^2 + mg = 0. \quad (25)$$

Grading: Maximum score is 4 points for this subproblem:

- 1.5 points for correct Lagrangian, where: 0.5 points for correct K , 0.5 points for correct V , and 0.5 points for combining K and V to find the Lagrangian.

- 1.5 points for Lagrange’s equation for r , where 0.5 points for $\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}})$, 0.5 points for correct $\frac{\partial L}{\partial r}$, and 0.5 points for combining them in to Lagrange’s equation for r .
- 1 point for Lagrange’s equation for ϕ , where: 0.5 points for noting that $\frac{\partial L}{\partial \phi} = 0$, and 0.5 points is for finding $\dot{\phi} = \frac{p_\phi}{mr^2}$, or the time derivative of this expression.

b) Reduce the equations of motion to a one dimensional problem in r .

Answer: In order to eliminate $\dot{\phi}$ from the equation of motion in Eq. (26) we use the cyclic property of the coordinate ϕ and write $\dot{\phi}$ it in terms of the constant generalised momentum p_ϕ as $\dot{\phi} = \frac{p_\phi}{mr^2}$. Using this we find the equation

$$2\ddot{r} - r \frac{p_\phi^2}{m^2 r^4} + g = 0$$

or

$$\ddot{r} = \frac{p_\phi^2}{2m^2 r^3} - \frac{g}{2}. \quad (26)$$

Grading: Maximum score is 1 point for this subproblem:

- 0.5 points for realising that $\dot{\phi}$ can be removed.
- 0.5 points for inserting the constant generalised momentum correctly.

c) Solve the equation of motion for r numerically and make a qualitative description of the motion.

Answer: In Fig. 2 we illustrate the motion of the rotating body through its position and velocity. We assume a mass $m = 1$ kg, angular momentum $p_\phi = 10$ kg m²/s, and with initial conditions $r(0) = 1$ m and $\dot{r}(0) = 0.5$ m/s. Code to reproduce this plot can be found in `oblig2.py`.

The equation of motion describes the competition between the first term on the right-hand side of Eq. (27), which acts like a centrifugal force, and the second term, which is the acceleration due to gravity. If $p_\phi = 0$, *i.e.* no angular momentum, we see that gravity will simply pull the rope down, shortening the distance r of the upper body to the hole.

If we instead have a nonzero p_ϕ , the upper body is rotating and there is a centrifugal term counteracting gravity. However, if the centrifugal term is smaller than the pull from gravity, *i.e.* if $p_\phi^2 < m^2 r^3 g$, gravity will still win out and pull the rope downwards. This will in turn cause the upper body to rotate faster (as $\dot{\phi} = p_\phi / mr^2$), which will increase the centrifugal force until it eventually wins out over gravity. This will cause the rope to start moving up again, which will weaken the centrifugal term until gravity wins out again, and the cycle is repeated.

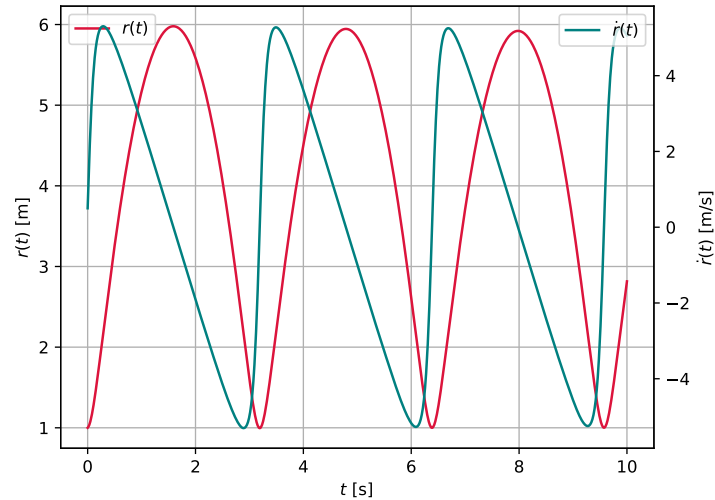


Figure 2: Position (red) and velocity (green) of the rotating body.

The result is an oscillatory behaviour of r . However, the nonlinearity of the $1/r^3$ term means that this oscillation is far from harmonic. If the initial conditions are such that r gets close to zero, the $1/r^3$ term will increase very rapidly, causing a very sharp change in the velocity of the particle. On the other hand, when the rope is long enough that gravity wins out, the acceleration is almost constant, giving a parabola-like motion. The result is a kind of motion which resembles a bouncing ball, with r moving in nice parabolas interrupted by sharp changes in velocity near $r = 0$.

Grading: Maximum score is 2 points for this subproblem:

- 1 point for a good plot of the numerical solution.
- 1 points for a good discussion of the general behaviour of the system.