

FYS3120 Classical Mechanics and
Electrodynamics

Solutions for Problem Set 4

February 17, 2025

Problem 1 Symmetries and Noether's theorem in use

We start from the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - gm x, \quad (1)$$

which describes a free falling mass m with gravity in the negative x direction, or any other potential linear in the position.

We remind you that the criterion for the Lagrangian to give form symmetry for the resulting equations of motion under a transformation is

$$L[q'(q, t), \dot{q}'(q, \dot{q}, t), t'(q, t)] \frac{dt'}{dt} = L[q, \dot{q}, t] + \frac{dF(q, t)}{dt}. \quad (2)$$

- a) Show why the equation of motion from this Lagrangian is symmetric under time translations

$$t \rightarrow t' = t + a, \quad x \rightarrow x' = x,$$

and use Noether's theorem to find what quantity is conserved.

Answer: Since there is not explicit time in L , and

$$\dot{x}' \equiv \frac{dx'}{dt'} = \frac{dt}{dt'} \frac{dx}{dt} = \dot{x}$$

we have

$$L[x', \dot{x}'] \frac{dt'}{dt} = L[x, \dot{x}]$$

thus the e.o.m. must have form symmetry. Noether's theorem gives that

$$\delta Q = - \left(\frac{\partial L}{\partial \dot{x}} \dot{x} - L \right) \delta a = -H \delta a$$

is conserved, where we have used an infinitesimal time translation δa . Since the translation parameter is arbitrary, the total energy $H = K + V$ is conserved.

Grading: Maximum score is 1 point for this subproblem. 0.5 points for arguing why we have symmetry for e.o.m., 0.5 point for using Noether's theorem to find H .

- b) Show why the equation of motion from this Lagrangian is symmetric under space translations

$$t \rightarrow t' = t, \quad x \rightarrow x' = x + a,$$

and use Noether's theorem to find what quantity is conserved. Find the physical meaning of the conserved quantity.

Answer: We have

$$\dot{x}' \equiv \frac{dx'}{dt'} = \frac{dx'}{dt} = \dot{x},$$

which gives

$$L[x', \dot{x}'] \frac{dt'}{dt} = \frac{1}{2} m \dot{x}^2 - gm(x + a) = L[x, \dot{x}] - gma.$$

We can trap the constant in a total derivative using $F = -gmt$ so that

$$L[x', \dot{x}'] \frac{dt'}{dt} = L[x, \dot{x}] + \frac{dF}{dt},$$

and we fulfil the conditions for symmetry.

Noether's theorem gives that

$$\delta Q = \frac{\partial L}{\partial \dot{x}} \delta a - \delta G = m\dot{x}\delta a - \delta G,$$

is conserved, where we have used an infinitesimal space translation δa . From our expression for F the corresponding δG to first order in the infinitesimals is $\delta G = -gmt\delta a$, so

$$\delta Q = m\dot{x}\delta a + gmt\delta a = (m\dot{x} + gmt)\delta a.$$

Since the translation parameter is arbitrary, $m\dot{x} + gmt$ or $\dot{x} + gt$ is conserved/constant. This is the momentum or velocity in the x -direction corrected by the acceleration due to gravity,

$$\dot{x} = k - gt,$$

where k is the velocity at $t = 0$.

Comment: we have here essentially solved the equation of motion using only symmetry arguments and never the e.o.m. directly!

Grading: Maximum score is 2 point for this subproblem. 1 points for arguing why we have symmetry for e.o.m., including 0.5 point for using the total-time derivative. 0.5 points for using Noether's theorem to find the conserved quantity, 0.5 points for a correct interpretation.

c) Show that the **Galilean transformation** of the coordinates

$$t \rightarrow t' = t, \quad x \rightarrow x' = x - vt, \quad (3)$$

where v is a constant velocity, is a symmetry of this Lagrangian and will give the same equations of motion.

Answer: Using that

$$\dot{x}' = \frac{dx'}{dt'} = \frac{d}{dt}(x - vt) = \dot{x} - v,$$

gives

$$\begin{aligned}
 L[x', \dot{x}'] \frac{dt'}{dt} &= \frac{1}{2} m \dot{x}'^2 - g m x' = \frac{1}{2} m (\dot{x} - v)^2 - g m (x - vt) \\
 &= \frac{1}{2} m \dot{x}^2 - g m x - m \dot{x} v + \frac{1}{2} m v^2 + g m v t \\
 &= L[x, \dot{x}] + \frac{dF(x, t)}{dt}
 \end{aligned}$$

where

$$F(x, t) = -m x v + \frac{1}{2} m v^2 t + \frac{1}{2} g m v t^2,$$

is a function of only the coordinate and time, so that the total time-derivative can be ignored in the action. The new coordinates must then lead to the same e.o.m.

Grading: 1 point for showing that the Lagrangian satisfies the symmetry criterion.

d) Find what is conserved under the Galilean transformation.

Answer: Noether's theorem gives

$$\delta Q \equiv -\frac{\partial L}{\partial \dot{x}} t \delta v - \delta G = -m \dot{x} t \delta v - \delta G$$

where we have used an infinitesimal velocity δv to parameterise the transformation.

Starting from our expression for F we can find the corresponding δG to first order in the infinitesimals

$$\delta G = -m x \delta v + \frac{1}{2} g m t^2 \delta v,$$

The conserved quantity is then

$$\delta Q = -m \dot{x} t \delta v + m x \delta v - \frac{1}{2} g m t^2 \delta v = (m x - m \dot{x} t - \frac{1}{2} g m t^2) \delta v.$$

Ignoring the parameter δv we see that $x - \dot{x} t - \frac{1}{2} g t^2$ is constant, which is just the initial position of the particle.

Grading: 0.5 points for inserting correctly in the Noether charge, 0.5 points for drawing the right conclusions about the contribution from the total time derivative.

e) What is the most general possible Lagrangian with one generalised coordinate q and no explicit time-dependence, $L(q, \dot{q})$, that is invariant under Galilean transformations and gives distinct equations of motion?

Hint: It could be wise to start from the criterion for symmetry in terms of infinitesimal transformations that was demonstrated in the lectures:

$$\sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \left(\frac{d}{dt} \delta q_i - \dot{q}_i \frac{d}{dt} \delta t \right) \right] + \frac{\partial L}{\partial t} \delta t + L \frac{d}{dt} \delta t = \frac{d}{dt} \delta G. \quad (4)$$

Answer: With the infinitesimal transformation

$$t \rightarrow t' = t + \delta t, \quad q \rightarrow q' = q + \delta q, \quad (5)$$

with $\delta t = 0$ and $\delta q = -t \delta v$, the symmetry criterion (4) gives

$$-\frac{\partial L}{\partial q} t \delta v + \frac{\partial L}{\partial \dot{q}} \left(\frac{d}{dt} (-t \delta v) \right) = \frac{dG(q, t)}{dt} \delta v, \quad (6)$$

where we have written δG as a function of the infinitesimal parameter δv and the coordinates q and t . Taking the derivatives and cancelling δv gives the relationship

$$-\frac{\partial L}{\partial q} t - \frac{\partial L}{\partial \dot{q}} = \frac{\partial G}{\partial q} \dot{q} + \frac{\partial G}{\partial t}. \quad (7)$$

Since the right-hand side is linear in $\dot{q}$, so must the left-hand side be. The most general possible Lagrangian is then a quadratic polynomial in $\dot{q}$

$$L(q, \dot{q}) = a(q) \dot{q}^2 + c(q),$$

with two unknown functions of q , a and c . Any first order term $b(q) \dot{q}$ in the Lagrangian could be removed before we started looking at the transformations because it is a total derivative term (does not give any different e.o.m.)

$$b(q) \dot{q} = \frac{d}{dt} B(q), \quad \text{where} \quad B'(q) \equiv \frac{dB}{dq} = b(q).$$

Then

$$\frac{\partial L}{\partial q} = \frac{\partial a}{\partial q} \dot{q}^2 + \frac{\partial c}{\partial q},$$

but since we do not have a term with $\dot{q}^2$ on the right-hand side we must have $\frac{\partial a}{\partial q} = 0$, which implies $a = c_1$ is a constant. Inserting the current form of the Lagrangian into the criterion we get

$$-\frac{\partial c}{\partial q} t - 2c_1 \dot{q} = \frac{\partial G}{\partial q} \dot{q} + \frac{\partial G}{\partial t}.$$

Since $\dot{q}$ is a free variable (the e.o.m. has not been imposed), we can pick out the $\dot{q}$ dependence on both sides to get the relationships.

$$-2c_1 = \frac{\partial G}{\partial q}, \quad -\frac{\partial c}{\partial q} t = \frac{\partial G}{\partial t}.$$

Since

$$\frac{\partial^2 G}{\partial q \partial t} = \frac{\partial^2 G}{\partial t \partial q},$$

we must have

$$\frac{\partial^2 c}{\partial q^2} = 0 \quad \Rightarrow \quad c(q) = c_2 q + c_3.$$

The most general Lagrangian is then

$$L(q, \dot{q}) = c_1 \dot{q}^2 + c_2 q,$$

where we have set $c_3 = 0$ since a constant can be removed from the Lagrangian without changing the e.o.m. This has the same form as the free fall Lagrangian (1).

Grading: Maximum score is 2 point for this subproblem. 0,5 points for writing down the symmetry criterion correctly for this case, 0.5 points for proposing the quadratic polynomial Lagrangian, 0.5 point for realising we can compare order by order in $\dot{q}$, 0.5 for working out the last relationships.

- f) Find all the symmetries of the free-fall Lagrangian in (1) for infinitesimal transformations $\delta x(x, t)$ and $\delta t(x, t)$ of the coordinates. *Hint:* We are looking for the most general expressions that $\delta x(x, t)$ and $\delta t(x, t)$ can take. It can be useful to compare powers of $\dot{x}$ when writing out (4).

Answer: We can do this by starting from the criterion for symmetries (4) and inserting the Lagrangian in (1) giving

$$\begin{aligned} \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \left(\frac{d}{dt} \delta x - \dot{x} \frac{d}{dt} \delta t \right) + \frac{\partial L}{\partial t} \delta t + L \frac{d}{dt} \delta t &= \frac{d}{dt} \delta G \\ -gm \delta x + m \dot{x} \left(\frac{d}{dt} \delta x - \dot{x} \frac{d}{dt} \delta t \right) + \left(\frac{1}{2} m \dot{x}^2 - gm x \right) \frac{d}{dt} \delta t &= \frac{d}{dt} \delta G \\ -gm \delta x + m \dot{x} \left(\frac{\partial \delta x}{\partial x} \dot{x} + \frac{\partial \delta x}{\partial t} - \frac{\partial \delta t}{\partial x} \dot{x}^2 - \frac{\partial \delta t}{\partial t} \dot{x} \right) \\ + \left(\frac{1}{2} m \dot{x}^2 - gm x \right) \left(\frac{\partial \delta t}{\partial x} \dot{x} + \frac{\partial \delta t}{\partial t} \right) &= \frac{\partial \delta G}{\partial x} \dot{x} + \frac{\partial \delta G}{\partial t} \end{aligned}$$

where we have used that $\delta x(x, t)$ and $\delta t(x, t)$ to expand the total time derivatives. We can now compare each power of $\dot{x}$ which must be the same on the two sides of the equation. This gives the four equations

$$\frac{\partial \delta t}{\partial x} = 0, \quad (\dot{x}^3) \quad (8)$$

$$\frac{\partial \delta x}{\partial x} - \frac{1}{2} \frac{\partial \delta t}{\partial t} = 0, \quad (\dot{x}^2) \quad (9)$$

$$m \frac{\partial \delta x}{\partial t} - mgx \frac{\partial \delta t}{\partial x} = \frac{\partial \delta G}{\partial x}, \quad (\dot{x}^1) \quad (10)$$

$$-gm \delta x - mgx \frac{\partial \delta t}{\partial t} = \frac{\partial \delta G}{\partial t}. \quad (\dot{x}^0) \quad (11)$$

We can now use these to find the most general $\delta x(x, t)$ and $\delta t(x, t)$. From (8) we know that δt is a function of t only, i.e. $\delta t(t) = a(t)$. From (9) we get that

$$\delta x(x, t) = \frac{1}{2}x \frac{da}{dt} + b(t),$$

where $b(t)$ is also a function of t only. Inserting these expression into (12) and (13) we get

$$m \left(\frac{1}{2}x \frac{d^2a}{dt^2} + \frac{db}{dt} \right) = \frac{\partial \delta G}{\partial x}, \quad (\dot{x}^1) \quad (12)$$

$$-gm \left(\frac{1}{2}x \frac{da}{dt} + b \right) - mgx \frac{da}{dt} = \frac{\partial \delta G}{\partial t}. \quad (\dot{x}^0) \quad (13)$$

Using that

$$\frac{\partial^2 \delta G}{\partial x \partial t} = \frac{\partial^2 \delta G}{\partial t \partial x},$$

then gives

$$\frac{1}{2}x \frac{d^3a}{dt^3} + \frac{d^2b}{dt^2} = -g \frac{3}{2} \frac{da}{dt}.$$

Since there is not other x dependence the first term on the left hand side means that

$$\frac{d^3a}{dt^3} = 0 \quad \Rightarrow \quad a(t) = c_1 t^2 + c_2 t + c_3.$$

Then

$$\frac{d^2b}{dt^2} = -g \frac{3}{2} \frac{da}{dt} = -g \frac{3}{2} (2c_1 t + c_2) = -3c_1 g t - \frac{3}{2} c_2 g,$$

which gives

$$b(t) = -\frac{1}{2} c_1 g t^3 - \frac{3}{4} c_2 g t^2 + c_4 t + c_5.$$

Thus, the most general symmetry of the Lagrangian in (1) is under the transformations

$$\begin{aligned} \delta t(t) &= c_1 t^2 + c_2 t + c_3, \\ \delta x(x, t) &= c_1 \left(xt - \frac{1}{2} g t^3 \right) + c_2 \left(\frac{1}{2} x - \frac{3}{4} g t^2 \right) + c_4 t + c_5. \end{aligned}$$

Just out of interest, the corresponding conserved quantity for the full symmetry are from Noether's theorem given by

$$\begin{aligned} &c_1 m \left(x \dot{x} t - \frac{1}{2} g \dot{x} t^3 - \frac{1}{2} \dot{x}^2 t^2 - \frac{1}{2} x^2 + \frac{1}{2} g x t^2 - \frac{1}{8} g^2 t^4 \right) \\ &+ c_2 m \left(\frac{1}{2} x \dot{x} - \frac{3}{4} g \dot{x} t^2 - \frac{1}{2} \dot{x}^2 t + \frac{1}{2} g x t - \frac{1}{4} g^2 t^3 \right) \\ &+ c_3 m \left(-\frac{1}{2} \dot{x}^2 - g x \right) + c_4 m \left(\dot{x} t - x + \frac{1}{2} g t^2 \right) + c_5 m (\dot{x} + g t) \end{aligned}$$

One can here recognise the conservation under space translation (factor of c_5), space translation (factor of c_3), and the Galilean transformation (factor of c_4).

Grading: Maximum score is 2 point for this subproblem. 0,5 points for writing down and expanding the symmetry criterion correctly, 0.5 point for comparing order by order in $\dot{x}$, up to 1 point for solving the differential equations.

Problem 2 The brachistochrone problem

Assume a small body, named P in Fig. 1, moves in a vertical plane under the influence of gravity only. It leaves a point A with zero velocity and follows (without friction) a given path in the plane which passes through a second point B , as shown in the figure. Assume the path between the two points A and B can be changed, while the points themselves stay fixed. For which path between the two points does the body spend the least time on the transit from point A to point B ?

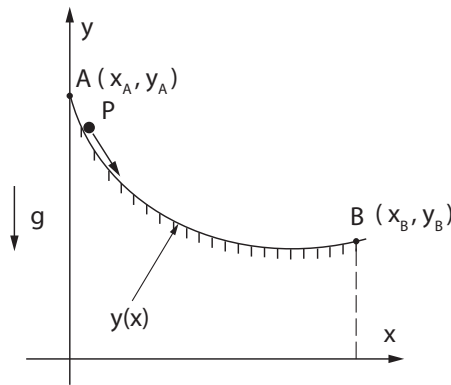


Figure 1: Illustration of the brachistochrone problem.

The challenge for you is the following: Find the solution to the brachistochrone problem by using the correspondence between the variational problem (finding the “path of shortest time”) and the Lagrangian of mechanics, in the way discussed in the lectures.

The body P is to be treated as a point particle of mass m and the path is represented by a function $y(x)$ with x as the horizontal and y as the vertical coordinate. The boundary conditions, which fix the positions of point A and B , are specified as, $y(x_A) = y_A$, $y(x_B) = y_B$. To simplify the equations assume in the following that the initial coordinates are $x_A = y_A = 0$.

- a) Show that the time T spent by the body on the way between A and B can be

expressed as an integral of the form

$$T[y(x)] = \int_{x_A}^{x_B} L(y, y') dx, \quad (14)$$

with $y' = \frac{dy}{dx}$, and with

$$L(y, y') = \sqrt{\frac{1 + y'^2}{-2gy}}. \quad (15)$$

Hint: Make use of energy conservation in the form $\frac{1}{2}mv^2 + mgy = 0$.

Answer: To find the total time we need to start from the infinitesimal time dt the particle takes to traverse an infinitesimal horizontal distance dx . We have the well known $dt = ds/v$. v here is a function of the vertical position y which can be found from energy conservation to be $v = \sqrt{-2gy}$. Since

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2} dx, \quad (16)$$

we now have

$$dt = \sqrt{\frac{1 + y'^2}{-2gy}}. \quad (17)$$

The total time is now the integral over these infinitesimal times in Eq. (14), which depends on the path $y(x)$.

Grading: Maximum score is 1 point for this subproblem:

- 1 point for a correct derivation of the action integral.

- b) In this case it is not so easy to find the equations of motion directly from Lagrange's equation, however, we can use other properties from analytical mechanics. With $L(y, y')$ interpreted as a Lagrangian (x then plays the role of t in the usual formulation) the canonically conjugate momentum is $p = \frac{\partial L}{\partial y'}$ and the Hamiltonian is $H = py' - L$. Explain why H is a constant of motion and use this fact to show that $y(x)$ satisfies a differential equation of the form

$$(1 + y'^2)y = -k^2, \quad (18)$$

with k as a constant.

Answer: We find an explicit expression for H in terms of position y and velocity to check "time" dependence:

$$H = \frac{\partial L}{\partial y'} y' - L = \frac{y'}{\sqrt{-2gy}\sqrt{1 + y'^2}} y' - \sqrt{\frac{1 + y'^2}{-2gy}} = -\frac{1}{\sqrt{(-2gy)(1 + y'^2)}}. \quad (19)$$

We see that there is no explicit “time” dependence $\frac{\partial H}{\partial x} = 0$, and since $\frac{\partial H}{\partial x} = \frac{dH}{dx}$, H must be time-independent and a constant of motion. Here we also accept answers that argue that the Lagrangian has no explicit “time” dependence.

Since H is constant, say with value W ,¹ we have

$$\begin{aligned} -\frac{1}{\sqrt{(-2gy)(1+y'^2)}} &= W \\ 1 &= W^2(-2gy)(1+y'^2) \\ (1+y'^2)y &= -\frac{1}{2gW^2} = -k^2, \end{aligned} \quad (20)$$

where $k = \sqrt{1/2gW^2}$.

Grading: Maximum score is 2 points for this subproblem:

- 1 point for the explanation of why H is a constant of motion arguing from x -dependence in either L or in H .
- 1 point for showing that $(1+y'^2)y = -k^2$.

c) Equation (18) has a solution which can be written on *parametric form* as

$$x(\theta) = \frac{1}{2}k^2(\theta - \sin \theta), \quad (21)$$

$$y(\theta) = \frac{1}{2}k^2(\cos \theta - 1), \quad (22)$$

where θ has been introduced as a curve parameter. Show that (21) and (22) constitute a solution of the differential equation (18) by changing the variable from x to θ in (18), and by using the above expression for $y(\theta)$. In what way are the boundary conditions taken care of by this solution?

Answer: To change variable in (18) we need to re-express the velocity y' in terms of θ :

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta} \frac{d\theta}{dx}. \quad (23)$$

Here

$$\frac{dx}{d\theta} = \frac{1}{2}k^2(1 - \cos \theta) = -y, \quad (24)$$

so

$$y' = -\frac{dy}{d\theta} \frac{1}{y}. \quad (25)$$

This means that we can write Eq. (18) in terms of θ as

$$(1+y'^2)y = \left(1 + \frac{1}{y^2} \left(\frac{dy}{d\theta}\right)^2\right)y = \left(y^2 + \left(\frac{dy}{d\theta}\right)^2\right)\frac{1}{y} = -k^2. \quad (26)$$

¹Be aware that H does not have units of energy since our Lagrangian has units s/m.

With

$$\frac{dy}{d\theta} = -\frac{1}{2}k^2 \sin \theta, \quad (27)$$

we can insert the solution $y(\theta)$

$$\begin{aligned} \left(y^2 + \left(\frac{dy}{d\theta} \right)^2 \right) \frac{1}{y} &= \left(\left(\frac{1}{2}k^2(\cos \theta - 1) \right)^2 + \left(\frac{1}{2}k^2 \sin \theta \right)^2 \right) \frac{1}{y} \\ &= \frac{1}{4}k^4((\cos \theta - 1)^2 + \sin^2 \theta) \frac{1}{y} \\ &= \frac{1}{4}k^4(2 - 2 \cos \theta) \frac{1}{y} \\ &= -k^2 y \frac{1}{y} = -k^2, \end{aligned} \quad (28)$$

which shows that (21) and (22) constitute a solution of the equation.

The first boundary condition is

$$x_A = 0 \quad \Rightarrow \quad 0 = \frac{1}{2}k^2(\theta_A - \sin \theta_A) \quad \Rightarrow \quad \theta_A = 0, \quad (29)$$

$$y_A = 0 \quad \Rightarrow \quad 0 = \frac{1}{2}k^2(\cos \theta_A - 1) \quad \Rightarrow \quad \theta_A = 2n\pi, \quad (30)$$

where $n = 0, 1, \dots$. We see that $\theta_A = 0$ fulfils this boundary condition in the new solution. The second boundary condition gives the equations

$$x_B = \frac{1}{2}k^2(\theta_B - \sin \theta_B), \quad (31)$$

$$y_B = \frac{1}{2}k^2(\cos \theta_B - 1), \quad (32)$$

which can be solved numerically for k and θ_B for a given x_B and y_B .

Grading: Maximum score is 2 points for this subproblem:

- 1 point for showing that the parameterisations of x and y give the solutions to equation 18.
- 1 point for explaining what the boundary conditions are in terms of θ (0.5 for θ_A and 0.5 for θ_B).