

FYS3120 Classical mechanics and electrodynamics
Midterm exam – Spring term 2021

March 10, 2021

Important information:

- Your answers are to be submitted electronically as pdf-files in Inspira, either generated from L^AT_EX or scanned, at the latest Friday 26th of March at 14.00 local Oslo time.
- This deadline is absolute, but you may submit multiple times. Only your last submission will be evaluated.
- This mid-term exam accounts for roughly 20% of the total grade in FYS3120.
- As this is a home-exam you are free to use any sources of information you may want, and you may collaborate with other students on solving the problems. However, the text of the submitted answers must be your own, and the usual rules of plagiarism apply. (We may check answers for similarities.)
- The best possible score on this exam is 20 points. Up to one point will be given for clear, concise and well presented answers, including appropriate figures and/or diagrams.
- You may give your answers either in English or Norwegian.
- Good luck!

Question 1 Is it a force? Is it geometry? No it's...

Prepare to get your mind blown through the terrible power of the Lagrangian.

- a) Consider the following Lagrangian with two degrees of freedom:

$$L(x, y, \dot{x}, \dot{y}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{\dot{y}^2}{kx^2}. \quad (1)$$

Find the coupled equations of motion and separate them. What sort of equation of motion is the separated equation for x ? Is this equation valid for $x = 0$? [2 points]

- b) Find the solutions of the equations of motion in **a** using $\omega = \sqrt{k/m}$. Comment on the role of the integration constants for y . [2 points]
- c) Let us now consider (1) as the Lagrangian of a free particle in the coordinates (x, y) with only kinetic terms. What is the metric g of this space? [1 point]
- d) Write down a Lagrangian for a particle with mass m moving in n dimensions, with no constraints and under a conservative force. Use the reasoning above to find an alternative Lagrangian formulated instead in terms of a free particle in $n + 1$ dimensions. Comment on the geometrical interpretation of this description. [2 points]

Question 2 Relativistic body in a central symmetric potential

In the following we will try to combine what we have learnt about motion in a central symmetric potential with special relativity. Start from the following relativistic Lagrangian for a body with mass m in a potential $V(\mathbf{r})$,

$$L = -\frac{mc^2}{\gamma} - V(\mathbf{r}), \quad (2)$$

where γ is the usual relativistic factor $\gamma = (1 - \beta^2)^{-1/2}$. Assume in all the following that the potential V is central symmetric.

- a) Show that the partial derivative of this Lagrangian with respect to *any* generalised velocity $\dot{q}$ is given by

$$\frac{\partial L}{\partial \dot{q}} = \frac{\gamma m}{2} \frac{\partial}{\partial \dot{q}} \mathbf{v}^2, \quad (3)$$

where $\mathbf{v}$ is the velocity of the body. [1 point]

- b) Show that the generalised momentum of the particle for Cartesian coordinates $\mathbf{r} = (x, y, z)$ is equal to the mechanical (relativistic) momentum. [1 point]

- c) Choose a good set of generalised coordinates (not Cartesian), find the Lagrangian in those coordinates, and identify any cyclic coordinates as well as the corresponding conserved quantity. [2 points]
- d) Argue that the motion must take place in a plane. [1 point]
- e) Since we know that the motion is in a plane, we can *instead* use polar coordinates (ρ, ϕ) , where ρ is the distance in that plane from the origin. Find the corresponding Lagrangian and the constant of motion ℓ associated with the Lagrangian symmetry $\phi \rightarrow \phi' = \phi + \delta\phi$. [1 point]
- f) Show that the Hamiltonian is conserved and find an expression for the conserved quantity in terms of the relativistic energy. [1 point]
- g) We want to determine the trajectory, or orbit equation, of the body $\rho(\phi)$. Find the equation of motion for ρ in terms of $\rho' = d\rho/d\phi$, $\rho'' = d^2\rho/d\phi^2$, and the constant of motion ℓ . [2 points]
- h) For central symmetric potentials (both non-relativistic and relativistic) it is common to make a change of variable of the type $\rho(\phi) = 1/u(\phi)$ to simplify the equation of motion. Find the equation of motion in terms of $u(\phi)$. [1 point]

Non-relativistic motion in a central symmetric potential is described by the *Binet equation*

$$u'' + u + \frac{m}{\ell^2} \frac{dV(\frac{1}{u})}{du} = 0, \quad (4)$$

which is identical to the answer in **h** in the limit $\gamma \rightarrow 1$ for the term proportional to γm .

- i) Assume that the potential is gravitational,

$$V(r) = -\frac{GmM}{r}, \quad (5)$$

where G is Newton's gravitational constant, and m and M are the masses of the two bodies. Find the difference in the orbits of two bodies between the relativistic and non-relativistic descriptions. If you could not solve **h** we encourage you to make an educated guess on the form of the equation. In the following you may assume that the mass M is so heavy that it can be considered at rest, and approximate γ to first order in v^2/c^2 , neglecting the radial component of the velocity v so that $v \simeq \rho\dot{\phi}$. *Hint:* It will be useful to describe the difference between the two models in terms of the (small) quantity

$$\epsilon = \frac{G^2 m^2 M^2}{\ell^2 c^2}. \quad (6)$$

[2 points]