

FYS3120 Classical mechanics and electrodynamics
Midterm exam – Spring term 2022

March 14, 2022

Important information:

- Your answers are to be submitted electronically as pdf-files in Inspira, either generated from L^AT_EX or scanned, at the latest Friday 25th of March at 14.00 local Oslo time.
- This mid-term exam accounts for roughly 20% of the total grade in FYS3120.
- As this is a home-exam you are free to use any sources of information you may want, and you may collaborate with other students on solving the problems. However, the text of the submitted answers must be your own, and the usual rules of plagiarism apply. (We may check answers for similarities.)
- The best possible score on this exam is 20 points. Up to one point will be given for clear, concise and well presented answers, including appropriate figures, plots and/or diagrams.
- If you find something you believe is unclear in these questions, you can ask for clarification by posting in the appropriate discussion thread on Canvas. Questions outside of Canvas will not be answered.
- You may give your answers either in English or Norwegian.
- Good luck!

Question 1 Metronomicon¹

In this exam we will be focusing on the motion of **metronomes**, consisting of a massless stiff rod with two masses m_1 and m_2 , rotating in the plane around a point P under the influence of gravity (and without friction). The masses are a distance r_1 and r_2 , respectively, from P , and we assume $r_1 m_1 > r_2 m_2$ for the proper functioning of the metronome.

At the attachment point the metronome has a torsion spring which acts just like a linear spring in storing potential energy, but as a function of the angle instead of the position, $V = \frac{1}{2}k\theta^2$, where $k > 0$ is the spring constant and θ is the angle of deflection away from the undeformed state, which for the metronome rod is the vertical position. The set-up is illustrated in Fig. 1.

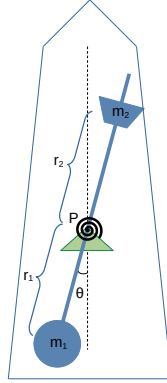


Figure 1: Illustration of a single metronome.

- a) Show that the Lagrangian for this system is

$$L(\theta, \dot{\theta}) = \frac{1}{2}I\dot{\theta}^2 + MRg \cos \theta - \frac{1}{2}k\theta^2, \quad (1)$$

where θ is the angle to the vertical for the rod, I the moment of inertia for the two masses for the rotation around P , $M = m_1 + m_2$ the total mass, and R the centre-of-mass distance from P . Justify the choices you make, such as for the coordinates. [2.5 points]

- b) Find the equation of motion for the system and solve it for the small angle approximation $\theta \ll 1$. [2 points]
- c) Identify any equilibrium solutions and interpret these physically. [2 points]
- d) Find Hamilton's equations and use them to plot the phase space of the system, including also large angles. Interpret the motion shown.

¹With thanks to Jan-Egil Ødegård for the original idea for this question.

Reasonable parameters for the metronome could for example be $m_1 = 0.16$ kg, $m_2 = 0.01$ kg, $r_1 = 0.02$ m, $r_2 = 0.16$ m. Consider different values of the spring constant k . *Hint:* For help plotting, see problem sets. [3 points]

We now place the metronome on a freely moving horizontal base of mass μ that moves horizontally without friction, *e.g.* by rolling.² The horizontal position of the centre-of-mass of the base is given as x . The metronome does not slide on the base. The set-up is illustrated in Fig. 2.

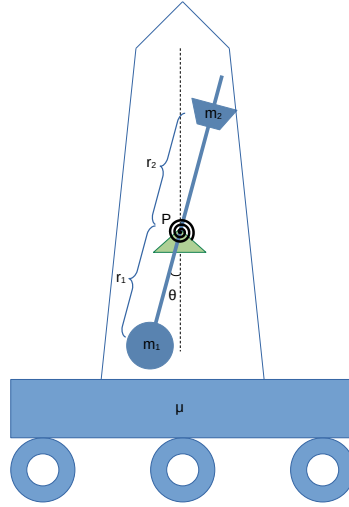


Figure 2: Illustration of a metronome on a freely moving base.

- e) Show that the Lagrangian for this system is

$$L = \frac{1}{2}(\mu + M)\dot{x}^2 + MR\dot{x} \cos \theta \dot{\theta} + \frac{1}{2}I\dot{\theta}^2 + MRg \cos \theta - \frac{1}{2}k\theta^2. \quad (2)$$

Hint: You may find use for the parallel axis theorem. [2.5 points]

- f) Solve analytically for the motion of the base in the x -direction in terms of θ . You may use that the base and the metronome is initially at rest at $t = 0$ when the metronome is started by releasing it from some initial angle. [2 points]
- g) Use Lagrange's equation to find the equation of motion for θ alone. [2 points]

We now want to generalize the above discussion to two identical metronomes sitting on the same base, as illustrated in Fig. 2.

² μ here does not indicate reduced mass, we just ran out of ms for the masses.

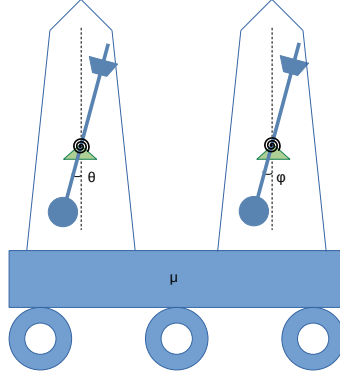


Figure 3: Illustration of two metronomes on a freely moving base.

- h) Use the small angle approximation to show that the equations of motion for the two metronomes are

$$(I - J)\ddot{\theta} + J(\theta\dot{\theta}^2 + \phi\dot{\phi}^2) - J\ddot{\phi} + (MRg + k)\theta = 0, \quad (3)$$

$$(I - J)\ddot{\phi} + J(\theta\dot{\theta}^2 + \phi\dot{\phi}^2) - J\ddot{\theta} + (MRg + k)\phi = 0, \quad (4)$$

where we have defined the constant

$$J = \frac{M^2 R^2}{\mu + 2M}. \quad (5)$$

Investigate what happens to the relative motion of the metronomes if they are released from rest with different starting angles. *Hint:* Try looking at the behaviour of $\psi = \theta - \phi$. [2 points]

- i) If instead of a torsion spring we use what is called an **escapement mechanism**, in practice a motor of sorts to drive the metronomes, energy is no longer conserved in the system. This can lead to new phenomena. The escapement mechanism can be modelled by a so-called **Van der Pol** term³

$$k(\theta^2 - \theta_0^2)\dot{\theta}, \quad (6)$$

where for small k the metronome is driven to oscillate with an amplitude of approximately $2\theta_0$.

Study the behaviour of two metronomes with an escapement mechanism numerically by replacing the spring terms in the equations of motion with Van der Pol terms. *Hint:* It would be wise to stay in the small angle approximation, using for example $\theta_0 = \phi_0 = 0.2$, with $k \sim 0.05$ Nms. The effect you are looking for is clearer the lighter the base is with respect to the metronomes. [1 point]

³This does not have a Lagrangian formulation.