

FYS3120 Classical mechanics and electrodynamics
Midterm exam – Spring term 2023

March 13, 2023

Important information:

- Your answers are to be submitted electronically as pdf-files in Inspira, either generated from L^AT_EX or scanned, at the latest Friday 24th of March at 14.00 local Oslo time.
- This mid-term exam accounts for roughly 20% of the total grade in FYS3120.
- As this is a home-exam you are free to use any sources of information you may want. However, the text of the submitted answers must be your own, and the usual rules of plagiarism apply to these. (We may check answers for similarities.)
- The best possible score on this exam is 20 points. Up to one point will be given for clear, concise and well presented answers, including appropriate figures, plots and/or diagrams.
- If you find something you believe is unclear in these questions, you can ask for clarification by posting in the appropriate discussion thread on Canvas. Questions outside of Canvas will not be answered.
- You may give your answers either in English or Norwegian. No code should be submitted.
- Good luck!

Question 1 Enbuchet, tobuchet, trebuchet

In this exam we will be focusing on the practical uses of **trebuchets** – ostensibly to throw cows in, or out, of castles.

We will start with a simple model of a throwing device consisting of a massless stiff rod mounted on a frictionless pivot point P some distance h above the ground, with a heavy counterweight mass m_1 on one end at a distance ℓ_1 from the pivot, and the projectile mass to be thrown, m_2 , at the other end, a distance ℓ_2 from the pivot. For simplicity we assume that the masses do not touch the ground at any point, *i.e.* $\ell_1, \ell_2 < h$, that we have a mechanism to release the mass m_2 at a given angle of the rod $\theta = \theta_f$, and that the counterweight and projectile are both point masses. See Fig. 1 for an illustration.

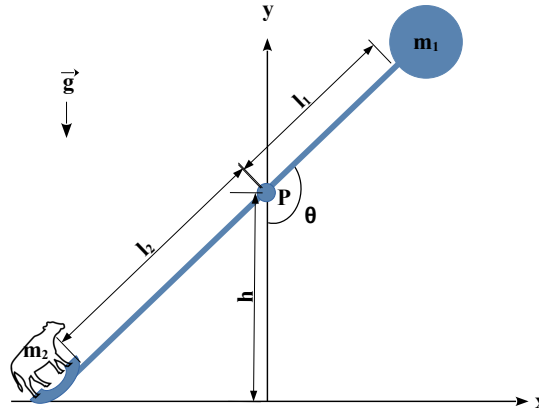


Figure 1: Illustration of a simple see-saw trebuchet model.

- Find the generalised coordinates and the Lagrangian for this system. [2 points]
- Find the equations of motion for this system from the Lagrangian. [2 points]
- Find the generalised momenta of the system. Is the angular momentum conserved? [1 points]
- Find the Hamiltonian and Hamilton's equations for this system. [2.5 points]
- Is energy conserved in this system? Give your argument in terms of the Lagrange–Hamilton formalism. [1 point]
- Plot the phase space of the system. For concreteness use the following parameters for the trebuchet: $m_1 = 7000$ kg, $m_2 = 15$ kg (mini-cow), $\ell_1 = 1.5$ m, and $\ell_2 = 10$ m.

Interpret the phase space plot in terms of the physical motion of a trebuchet when the counterweight mass m_1 starts from rest at some angle θ_i , and the projectile mass m_2 is released at some angle θ_f . Comment on the angle of the projectile mass when released and its velocity. *Hint:* For help with plotting, please see the problem sets. [2.5 points]

- g) It is (relatively) easy to show that – ignoring air resistance and the initial height of the projectile – a throw at an angle α from the horizontal with velocity v will reach a length of

$$L = \frac{v^2 \sin 2\alpha}{g}. \quad (1)$$

For a fixed velocity the optimal throwing angle for maximising the length is then $\alpha = \pi/4$. (You do not need to prove this.) Show that with this choice of release angle the maximum possible length of a trebuchet shot, if *all* the stored potential energy of the counterweight is transferred into the projectile kinetic energy,¹ is

$$L_{\max} = \frac{2m_1}{m_2} \ell_1 (1 - \cos \theta_i), \quad (2)$$

where θ_i is the starting position of the system. [1 point]

- h) We define the **range efficiency** as $\epsilon_L = L/L_{\max}$ where L is the length thrown. Solve Hamilton's equations numerically to determine the efficiency of this model for different release angles θ_f if the counterweight is released from rest at $\theta_i = \frac{5\pi}{6}$. Here we expect a plot. Find the maximum efficiency and the corresponding angle. [2 points]

We now add a hinge to the attachment of the counterweight m_1 and attach the counterweight on stiff rod a distance ℓ_3 from the hinge. There is no friction in the hinge. This set-up is illustrated in Fig. 2.

- i) Find the Lagrangian for this system. *Hint:* $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$. [2 points]
- j) Compare the range efficiency of the two trebuchet models by solving the equations of motion numerically. Assume again that the counterweight is released from rest at $\theta_i = \frac{5\pi}{6}$, and assume that the attachment rod starts vertically, *i.e.* that $\theta_i + \phi_i = \pi$. For the length of the counterweight rod assume $\ell_3 = 2$ m. *Hint:* In this and the following problem it is perfectly acceptable to use a computer algebra system to aid in the equation solving. However, it is your responsibility to describe clearly what you are doing in your answer. [2 points]

¹Note that we are *not* saying it is possible to transfer all the kinetic energy into the projectile like this, it is a hypothetical maximum.

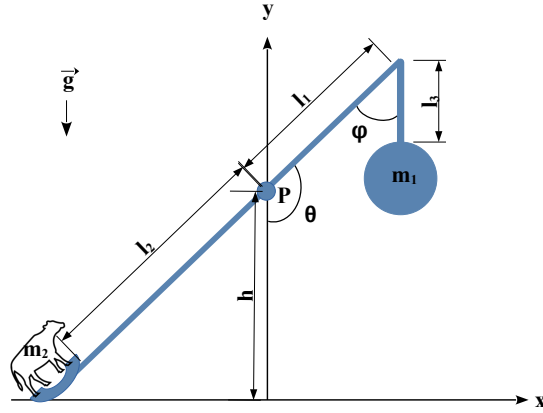


Figure 2: The see-saw model of a trebuchet with a hinged counterweight.

Finally, we attach a sling at the projectile end of the system with a length ℓ_4 to make a third trebuchet model. Assume again that the sling moves freely (it does not touch the ground at any point),² that the sling is stiff and the attachment has no friction. This is illustrated in Fig. 3.

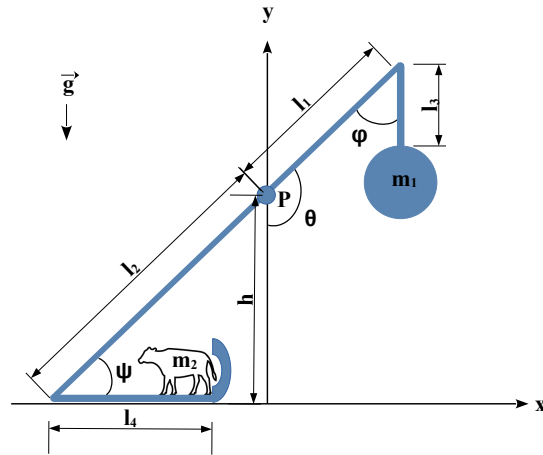


Figure 3: Trebuchet with a sling and a hinged counterweight.

- k) Compare the efficiency of this model with the previous two. For the initial conditions assume as above, and assume that the sling starts horizontally, *i.e.* that $\psi_i = \theta_i - \pi/2$, and at rest. For the length of the sling, use $\ell_4 = 6$ m. [1 point]

Fetchez la vache!

²This is an approximation to a real trebuchet where the sling is constrained to move along the ground before the projectile end is lifted into the air by the counterweight.