

FYS3120 Classical mechanics and electrodynamics
Midterm exam – Spring term 2024

March 5, 2024

Important information:

- Your answers are to be submitted electronically as pdf-files in Inspira, either generated from L^AT_EX or scanned, at the latest Friday 22nd of March at 14.00 local Oslo time.
- This mid-term exam accounts for roughly 20% of the total grade in FYS3120.
- As this is a home-exam you are free to use any sources of information you may want. However, the text of the submitted answers must be your own, and all the usual rules of plagiarism apply to these. (We will check answers for similarities.)
- You are required to include a short statement on your use of sources and tools, e.g. traditional sources (books etc.), online sources, numerical or analytic calculational software, and AI such as ChatGP, at the end of your answer.
- The best possible score on this exam is 20 points. Up to one point will be given for clear, concise and well presented answers, including appropriate figures, plots and/or diagrams.
- If you find something you believe is unclear in these questions, you can ask for clarification by posting in the appropriate discussion thread on the course Canvas page. Questions outside of Canvas will not be answered.
- You may give your answers either in English or Norwegian. No code should be submitted.
- Good luck!

Question 1 String theory

In this midterm exam we will focus on string theory. Or spring theory, given the time of year? We are not quite sure. Maybe you can figure that out.¹

Consider first a mechanical system consisting of two springs. See Fig. 1 for an illustration. We assume that the springs oscillate in the ϕ -direction only, they do *not* move in the x -direction. Assume further that each spring is an ideal spring with spring constant k and equilibrium length ℓ . A mass m is attached to the end of each spring.

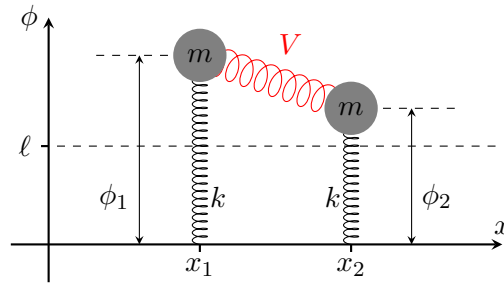


Figure 1: System of two springs in the $x\phi$ -plane.

- a) If there is no interaction between the springs, show that the Lagrangian becomes

$$L_0 = \frac{1}{2}m\dot{\phi}_1^2 + \frac{1}{2}m\dot{\phi}_2^2 - \frac{1}{2}k(\phi_1 - \ell)^2 - \frac{1}{2}k(\phi_2 - \ell)^2, \quad (1)$$

and give a brief description of the motion of the system. [2.5 points]

- b) Let $V_\phi = V_\phi(\phi_1, \phi_2)$ denote the potential for an interaction between the two springs.² Assume the interaction is symmetric between the springs, and has a stable equilibrium for $\phi_1 = \phi_2$, *i.e.* when the masses are at the same height. Explain why we can use the approximation

$$V_\phi = \frac{1}{2}\alpha(\phi_1 - \phi_2)^2, \quad (2)$$

near the equilibrium, where $\alpha > 0$ is the characteristic strength of the interaction. *Hint:* How should V behave as a function of $\phi_1 - \phi_2$?

[2 points]

¹Some of the blame for this exam should be directed at Carl Martin Fevang and Nils Johannes Mikkelsen who first worked out the initial version of this problem and made the nice figures.

²We will return to the interaction as a function of distance in the x -direction later.

- c) Use the Lagrange equations to find the equations of motion for ϕ_1 and ϕ_2 for the Lagrangian $L = L_0 - V_\phi$. You do not need to solve them. [2 points]

We now want to extend the spring system to include a set of N springs as shown in Fig. 2. We assume that interactions of the form (2) only exist between spring neighbours.

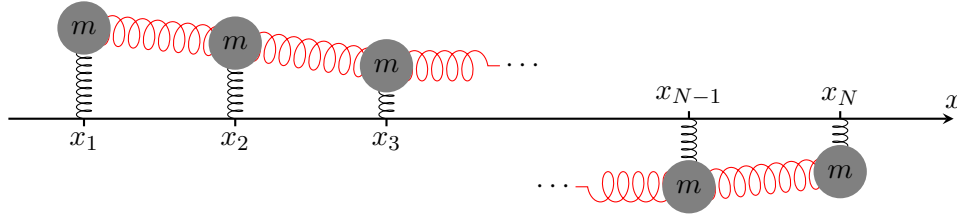


Figure 2: System of N interacting springs.

- d) Explain why we are allowed to set $\ell = 0$ in the following to simplify our calculations. Is this a symmetry of the system? [1 point]
- e) Determine the Lagrangian and find the equations of motion for the springs. Give both the total Lagrangian for the whole system and the contributions for a single spring. *Hint:* Introduce dummy variables ϕ_0 and ϕ_{N+1} , where $\phi_0 = \phi_1$ and $\phi_{N+1} = \phi_N$. [2.5 points]

We now want to take the limit of an infinite number of springs along the x -axis, including springs for negative N , and letting $N \rightarrow \infty$. Assume that the distance between the springs $\Delta x = x_{i+1} - x_i$ is constant.

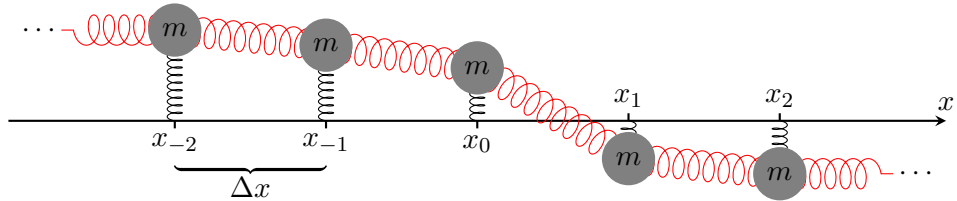


Figure 3: An infinite system of interacting springs.

- f) Perform a Legendre transformation to determine the Hamiltonian, H , for the entire (infinite) spring system. In the following we want you to use π_i for the generalised momenta of spring i . [1 point]

- g) Find Hamilton's equations and use this to numerically investigate the behaviour of the spring system. Describe the motion of the springs as a function of time. You may attach plots of the spring positions at various time-steps to illustrate your point. What is the effect of changing α ?

You do not need to use an infinite number of springs, choosing 15 points x_i with $i \in \{1, \dots, 15\}$ should be close enough. Start with the constants set to $m = 0.1 \text{ kg}$, $k = 2 \text{ kg/s}^2$, and $\alpha = 4 \text{ kg/s}^2$, and use the initial conditions $\phi_i = 0$ and $\pi_i = 0$ for all i , except $\pi_1 = 0.1$. Solving for a few seconds worth of time-steps should be sufficient. [2 points]

- h) What can you say about the total energy of the system? By requiring that the total energy is finite, find the resulting limitations for the generalised coordinates and their momenta as $x \rightarrow \pm\infty$. [2 points]

We now wish to take the continuum limit $\Delta x \rightarrow 0$ where the springs become a continuous system. To do so we have to check what happens to the interaction term in this limit.

- i) We now introduce a contribution to the interaction between neighbouring springs that depends on the relative distance in x , Δx , as the potential

$$V_x(\Delta x) = \frac{1}{2}\alpha\Delta x^2, \quad (3)$$

which has the same form as the potential in (2) for the same reasons. Explain why this has no effect on the motion of the springs if the strength of the interaction α is proportional to $1/(\Delta x)^2$, and discuss the implications for the total interaction potential $V = V_\phi + V_x$. [1 point]

- j) Assume the Lagrangian has physical units of energy (as usual). Determine the physical units of the constant $c^2 \equiv \frac{\alpha}{m}(\Delta x)^2$. *Hint:* Use $[L] = [\frac{1}{2}m\dot{\phi}^2]$ to find the units of ϕ , and use $[L] = [V]$ to find the units of α . [1 point]

- k) Let φ be a function that extends ϕ_i so that $\varphi(x_i, t) = \phi_i(t)$. Assume that α is proportional to $1/(\Delta x)^2$ and show that after taking the limit $\Delta x \rightarrow 0$, the equations of motion for ϕ_i can be written as the differential equation

$$(\square + M^2)\varphi(x, t) = 0, \quad (4)$$

where $\square$ is the d'Alembertian differential operator in one space dimension given by

$$\square = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}. \quad (5)$$

Find M^2 .

Equation (4) is often called the **Klein–Gordon equation** for scalar fields. [2 points]

- 1) Make a brief summary of any sources and tools you have used in your work with this exam. This should include what you have used the tools for and how you have utilised them. This extends to all online sources consulted, computational tools and AI systems. You need not list the course material (book, formulae collection, lecture notes, slides) here.