

Question 1 Spring-time

A mass m glides without friction along a stiff massless rod that rotates with constant angular velocity ω in the horizontal plane. The mass is further connected to the centre of rotation P by a spring which results in the force $\mathbf{F} = -2k\mathbf{r}$ acting on the mass, where $\mathbf{r}$ is the distance of the mass from the centre of rotation.

- a) Draw a figure for the system. [1 point]
- b) How many degrees of freedom does this system have? Choose generalised coordinate(s). [1 point]
- c) Show that the Lagrangian for this problem can be written as

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m\omega^2 r^2 - kr^2. \quad (1)$$

[2 points]

- d) Find the Lagrange equation(s). [1.5 points]
- e) Find the path that the mass takes. Discuss the solutions in terms of the constant

$$\omega_0^2 = \omega^2 - \frac{2k}{m},$$

and comment on their physical meaning. Are there any equilibrium solutions? [2.5 points]

- f) Find the Hamiltonian for this system. [1.5 points]
- g) Is the Hamiltonian conserved? If so, what is conserved? [1 point]
- h) Find Hamilton's equations. [1 point]
- i) Describe the phase space of the system. [1.5 points]
- j) From the criterion of form symmetry for the equations of motion

$$L[q', \dot{q}', t'] \frac{dt'}{dt} = L[q, \dot{q}, t] + \frac{dF(q, t)}{dt}, \quad (2)$$

show that the following transformation of the coordinates,

$$t \rightarrow t' = t, \quad r \rightarrow r' = r + \delta r = r + r_0 e^{\omega_0 t}, \quad (3)$$

where r_0 is a small quantity that parametrises the transformation, is a symmetry of our Lagrangian in Eq. (1). If it worries you, you may assume that $\omega_0^2 > 0$, although this is strictly not necessary. [2 points]

- k) Find the conserved quantity for the symmetry transformation in Eq. (3). [1.5 points]

Hint: In the lectures we showed that under an infinitesimal symmetry transformation of the coordinates

$$t \rightarrow t'(q, t) = t + \delta t(q, t), \quad q_i \rightarrow q'_i(q, t) = q_i + \delta q_i(q, t),$$

the corresponding conserved **Noether charge** is given by

$$\delta Q \equiv \sum_i \frac{\partial L}{\partial \dot{q}_i} \delta q_i - \left(\sum_i \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L \right) \delta t - \delta G, \quad (4)$$

where $\delta G = F(\delta q, \delta t)$ is an expression for $F(q, t)$ in Eq. (2) to first order in the infinitesimals.

- l) Find all the symmetries of the Lagrangian in Eq. (1). [2.5 points]

Hint: In the lectures we derived the following expression

$$\sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \left(\frac{d}{dt} \delta q_i - \dot{q}_i \frac{d}{dt} \delta t \right) \right] + \frac{\partial L}{\partial t} \delta t + L \frac{d}{dt} \delta t = \frac{d}{dt} \delta G, \quad (5)$$

that must be fulfilled for all (off-shell) symmetry transformations of the Lagrangian. We remind you that total time-derivatives can be written in terms of partial derivatives as for example

$$\frac{d}{dt} \delta G(q, t) = \sum_i \frac{\partial \delta G}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial \delta G}{\partial t} = \sum_i \frac{\partial \delta G}{\partial q_i} \dot{q}_i + \frac{\partial \delta G}{\partial t}.$$