

FYS3120 Classical Mechanics and
Electrodynamics

Solutions to Problem set 12

Deadline for handing in (in canvas): **12 May**, 3pm

Problem 1 Complex scalar fields at the bottom of a wine bottle

Let us begin by considering two real scalar fields $\phi_1(x)$ and $\phi_2(x)$ with the same mass m , described by the Lagrangian density¹

$$\mathcal{L}(\phi_1, \phi_2) = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{1}{2}m^2 \phi_1^2 - \frac{1}{2}m^2 \phi_2^2. \quad (1)$$

- a) Show that this Lagrangian can equivalently be written in terms of a single complex scalar field

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2), \quad (2)$$

as

$$\mathcal{L}(\Phi) = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - m^2 \Phi^* \Phi. \quad (3)$$

- b) Show that the Lagrangian (3) is invariant under a continuous *global* transformation of the field

$$\Phi \rightarrow \Phi' = e^{iq\alpha} \Phi,$$

with a real parameter α , and where q is the charge of Φ .² That the transformation is global means that α is independent of x^μ , if $\alpha(x)$ then the transformation is *local*.

- c) Show that the conserved current associated with the symmetry transformation in **b)** is

$$J^\mu = iq(\Phi \partial^\mu \Phi^* - \Phi^* \partial^\mu \Phi), \quad (4)$$

if α is small, and find the Noether charge that is conserved. You can use that Φ is normalised so that³

$$\int i \left(\Phi \frac{\partial \Phi^*}{\partial t} - \Phi^* \frac{\partial \Phi}{\partial t} \right) d^3r = 1. \quad (5)$$

We will now introduce an extra term in the Lagrangian so that we have a potential

$$V(\Phi) = -\mu^2 |\Phi|^2 + \lambda |\Phi|^4, \quad (6)$$

where $\mu^2 > 0$ and $\lambda > 0$. This is known as the Mexican hat, or wine-bottle potential and is illustrated in Fig. 1. Sometimes it is even called the (abelian) Higgs potential. The total Lagrangian is⁴

$$\mathcal{L}(\Phi) = (\partial_\mu \Phi)(\partial^\mu \Phi^*) + \mu^2 \Phi \Phi^* - \lambda (\Phi \Phi^*)^2. \quad (7)$$

¹For simplicity we will be working in natural units where $\hbar = c = 1$.

²The importance of the charge here will become more clear later.

³This is the standard normalisation for the probability density of a Klein-Gordon wave function in relativistic quantum mechanics.

⁴Notice here that the mass term $\mu^2 \Phi \Phi^*$ now has the “wrong” sign compared to the standard Klein-Gordon field.

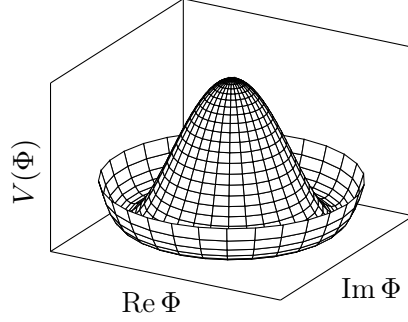


Figure 1: Mexican hat or wine-bottle potential $V(\Phi) = -\mu^2|\Phi|^2 + \lambda|\Phi|^4$.

- d) Show that $|\Phi|^2 = \frac{\mu^2}{2\lambda}$ is the degenerate minimum of the potential $V(\Phi)$.
- e) Find three different symmetries of the Lagrangian density in Eq. (7).
Hint: We are not looking for a lot of calculations.
- f) Expand the Lagrangian density around the minimum where $\Phi_0 = \sqrt{\frac{\mu^2}{2\lambda}}$ with a field defined by

$$\tilde{\Phi}(x) \equiv \Phi(x) - \Phi_0(x) = \Phi(x) - \sqrt{\frac{\mu^2}{2\lambda}},$$

and write the Lagrangian density in terms of this field as

$$\mathcal{L}(\tilde{\Phi}) = (\partial_\mu \tilde{\Phi})(\partial^\mu \tilde{\Phi}^*) - \frac{\mu^2}{2}(\tilde{\Phi} + \tilde{\Phi}^*)^2 - \sqrt{2\mu^2\lambda}\tilde{\Phi}\tilde{\Phi}^*(\tilde{\Phi} + \tilde{\Phi}^*) - \lambda(\tilde{\Phi}\tilde{\Phi}^*)^2. \quad (8)$$

- g) What are the symmetries of the Lagrangian density for $\tilde{\Phi}$ compared to the symmetries of Φ ?
- h) Show that the Lagrangian in (8) can in turn be rewritten in the form of a massive real scalar field h with mass $m_h = \sqrt{2\mu^2}$, and a massless real scalar field ϕ . *Hint:* Decompose $\tilde{\Phi} = \frac{1}{\sqrt{2}}(h + i\phi)$.
- i) Returning to the field transformation in question b), if we instead use a local transformation parameter $\alpha(x)$ and add an electromagnetic field $A^\mu(x)$ to the Lagrangian in Eq. (3), that couples to the complex scalar field with additional Lagrangian terms

$$\mathcal{L}(A^\mu) = -\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} - A^\mu J_\mu + q^2 A_\mu A^\mu \Phi^* \Phi, \quad (9)$$

where J_μ is the conserved Noether current we found in c) and q is still

the charge of the field Φ ,⁵ show that the total Lagrangian

$$\mathcal{L}(\Phi, A^\mu) = (\partial_\mu \Phi^*)(\partial^\mu \Phi) - V(\Phi) - \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu} - A^\mu J_\mu + q^2 A_\mu A^\mu \Phi^* \Phi, \quad (10)$$

is invariant under the transformation if A^μ transforms under the gauge transformation

$$A^\mu(x) \rightarrow A'^\mu(x) = A^\mu(x) + \partial^\mu \alpha(x).$$

Hint: The easiest way to do this is to define a new **covariant derivative**

$$D_\mu \equiv \partial_\mu - iqA_\mu, \quad (11)$$

and use this to show that the total Lagrangian can be written

$$\mathcal{L}(\Phi, A^\mu) = (D_\mu \Phi)^*(D^\mu \Phi) - V(\Phi) - \frac{1}{4\mu_0} F^{\mu\nu} F_{\mu\nu}, \quad (12)$$

and then study how $D_\mu \Phi$ transforms.

⁵We see here that q indeed sets the strength of the interaction between Φ and A_μ , so its interpretation as charge is valid.