

FYS3120 Classical Mechanics and
Electrodynamics

Problem set 2

Deadline for handing in (in canvas): **10 February**, 3pm

Problem 1 Reduced mass

Let us look at the generic two-body problem with two objects of mass m_1 and m_2 . Assume that the potential (force) between the two bodies is only dependent on the distance between the two objects – a so-called **central potential** – as would be the case for both gravitational and electrostatic potentials (forces). There are no constraints.

- a) Write down the Lagrangian L in terms of the Cartesian coordinates $\mathbf{r}_1$ and $\mathbf{r}_2$ of the two objects.
- b) Make the change of variables to the (generalised) coordinates $(\mathbf{r}, \mathbf{R})$, where

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \quad (1)$$

$$\mathbf{R} = \frac{m_1}{m_1 + m_2} \mathbf{r}_1 + \frac{m_2}{m_1 + m_2} \mathbf{r}_2, \quad (2)$$

$\mathbf{R}$ being the **centre-of-mass** of the two bodies, and show that the resulting Lagrangian is

$$L = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(r), \quad (3)$$

where $M = m_1 + m_2$ is the **total mass**, and

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \quad (4)$$

is the **reduced mass**.

- c) Find Lagrange's equations in the new coordinates and show that the solution for the path of $\mathbf{R}$ is

$$\mathbf{R}(t) = \mathbf{R}_0 + \mathbf{v}_0 t, \quad (5)$$

where $\mathbf{R}_0$ is some initial position at $t = 0$ and $\mathbf{v}_0$ is the initial velocity.

- d) What is the physical interpretation of the equations of motion? Argue why it is sufficient to study only the Lagrangian

$$L = \frac{1}{2} \mu \dot{\mathbf{r}}^2 - V(r), \quad (6)$$

in the two-body problem.

- e) Express the Lagrangian in Eq. (6) in spherical coordinates, where $\mathbf{r} = (x, y, z)$ with

$$x = r \cos \phi \sin \theta,$$

$$y = r \sin \phi \sin \theta,$$

$$z = r \cos \theta.$$

Use this to show that the angular momentum in the z -direction, ℓ_z , is conserved.

- e) In fact, since the choice of the z -axis in this problem is arbitrary, all components of angular momentum must be conserved.¹ This means in turn that the motion in the coordinate $\mathbf{r}$ must be in the plane perpendicular to the angular momentum – motion out of this plane would generate an angular momentum component in a different direction.

Choose the z -axis to point in the direction of angular momentum. Use this to show that the equation of the motion of the whole system can be reduced to the single differential equation

$$\mu\ddot{r} + \frac{\partial V_{\text{eff}}}{\partial r} = 0, \quad (7)$$

with the **effective potential** V_{eff} given by

$$V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2\mu r^2}, \quad (8)$$

where ℓ is the angular momentum.

Problem 2 Two bodies with the same mass, m , are connected with a massless rope through a small hole in a smooth horizontal plane. This is illustrated in Fig. 1. One body is moving on the plane, the other one is hanging at the end of the rope and can move vertically. At all instances the rope is tight. The acceleration due to gravity is g .

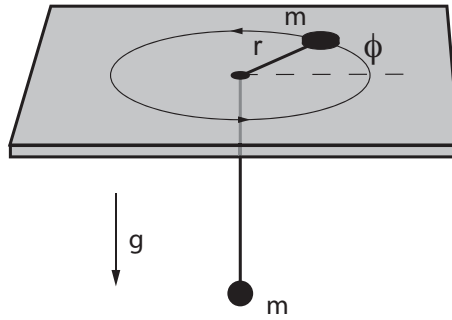


Figure 1: Two bodies connected by a rope through a small hole.

- Find Lagrange's equations of motion in polar coordinates (r, ϕ) .
- Reduce the equations of motion to a one dimensional problem in r .
- Solve the equation of motion for r numerically and make a qualitative description of the motion.

¹We shall show this more formally for central potentials later in the course.

Advice on numerical solutions

We take the opportunity to make some general suggestions for numerically solving differential equations which will be useful throughout the course:

- While it is extremely useful as a learning experience to have at some point implemented one's own routines for numerical solutions, you should approach hard problems with standard state-of-the-art solvers first. A lot of work has been put into these by rather clever people, and they are both resource efficient and very powerful.
- The standard way to numerically solve differential equations is to transform them to (coupled) sets of first order differential equations. This is an algebra job you need to do. (Or use a symbolic solver to do.) We shall later see a way of arriving at only first order differential equations when using the Hamiltonian formalism.
- In for example python the `scipy` package `solve_ivp` solver,² is a well tested standard solver with many available solution strategies. This takes as input a function that calculates a vector of the first order derivatives of the dependent variables, from the dependent variables themselves and the independent variable (usually time).

As an example, if you have a second order differential equation of the form (here time is the independent variable)

$$\ddot{r} = f(r)\dot{r} + g(r),$$

where f and g are some expressions with r , you define a new intermediate variable $s = \dot{r}$ so that you have two first order differential equations

$$\dot{r} = s, \tag{9}$$

$$\dot{s} = f(r)s + g(r). \tag{10}$$

Your job is then to write a function that calculates $\dot{r}$ and $\dot{s}$ from r and s (and potentially t) input and provide this to `solve_ivp`.

- If you have coupled second order differential equations you need to rewrite the equations so that only one second derivative appears per equation. This is done by substitution. The equations are then reduced to first order in the same way as above.
- Do not blindly trust numerical solutions. These should also be tested for reasonableness. For example:

²https://docs.scipy.org/doc/scipy/reference/generated/scipy.integrate.solve_ivp.html

- Is the long term behaviour of the system sensible?
- Is the problem numerically stable, meaning, if you change the initial conditions a little, does the solution vary wildly? (Should it be stable? Not all problems are, some demonstrate **chaotic behaviour**, where small changes in the inputs give dramatic changes in the results.)
- Do you expect energy or other quantities to be conserved from what you have learnt in the course, and if so, are they? At least reasonably well compared to the expected relative error of the solver? (Typically a parameter in the solver.)
- Beware also of running solutions for more time-steps than what your model should be good for. This may lead to unphysical behaviour, for example by violating boundary conditions, or simplifying assumptions such as the smallness of a quantity.