

FYS3120 Classical Mechanics and
Electrodynamics

Problem set 3

Deadline for handing in (in canvas): **17 February**, 3pm

Problem 1 Kepler problem

Let us return to the two-body problem from Problem Set 2 with two massive objects with masses m_1 and m_2 in a central potential of the form

$$V(r) = -\frac{k}{r}, \quad (1)$$

where r is the relative distance between the masses and $k > 0$. This constitutes what we call a **Kepler problem**, and with the right choice of k for the strength of the interaction would apply to both Coloumb and gravitational potentials for example.

We saw in Problem Set 2 that we can describe the motion of the objects by a Lagrangian given in terms of the relative position $\mathbf{r}$ as the generalised coordinate

$$L(\mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2}\mu\dot{\mathbf{r}}^2 + \frac{k}{|\mathbf{r}|}, \quad (2)$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2},$$

is the reduced mass.

- a) Show that the Hamiltonian of the system can be written

$$H(\mathbf{r}, \mathbf{p}) = \frac{1}{2\mu}\mathbf{p}^2 - \frac{k}{|\mathbf{r}|},$$

where $\mathbf{p}$ is the generalised momentum.

- b) Find Hamilton's equations of motion expressed in terms of $\dot{\mathbf{r}}$ and $\dot{\mathbf{p}}$.
- c) In Problem Set 2 we learned that angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is conserved for central potentials. Find one more conserved quantity in this problem and describe what is conserved.

The **Laplace-Runge-Lenz vector**¹ can be useful when studying two-body problems with potentials of the form (1). It is defined as

$$\mathbf{A} \equiv \mathbf{p} \times \mathbf{L} - \mu k \hat{\mathbf{r}}. \quad (3)$$

- d) Show that the Laplace-Runge-Lenz vector is a conserved quantity by computing $\dot{\mathbf{A}} = \frac{d\mathbf{A}}{dt}$.²

Hint: Rewrite the vector as

$$\mathbf{A} = \left(|\mathbf{p}|^2 - \frac{\mu k}{|\mathbf{r}|} \right) \mathbf{r} - (\mathbf{p} \cdot \mathbf{r})\mathbf{p}. \quad (4)$$

¹As usual, none of these people actually discovered it.

²You are not allowed to go back and use this in c).

You can then use the equations of motion found above for $\dot{\mathbf{r}}$ and $\dot{\mathbf{p}}$.

Warning: We need to be careful when differentiating $|\mathbf{r}|$ since this contains a vector. This is why we consistently use $|\mathbf{r}|$ rather than r in this problem.

- e) Since the motion must lie in a plane (see Problem Set 2), we can decompose the relative position in two orthogonal unit vectors $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$ and $\hat{\theta}$ for polar coordinates. Show that this lets us write the Laplace-Runge-Lenz vector as

$$\mathbf{A} = (|\mathbf{r}|p_\theta^2 - \mu k) \hat{\mathbf{r}} - |\mathbf{r}|p_\theta p_r \hat{\theta}, \quad (5)$$

where we have decomposed

$$\mathbf{p} = p_r \hat{\mathbf{r}} + p_\theta \hat{\theta}.$$

- f) If $\mathbf{A} = 0$ initially, what does this tell us about the motion of the bodies in light of the decomposition above?

Problem 2 In this exercise we shall consider a mass m attached to an ideal spring on a frictionless surface as an example of how to include dissipating (non-conserving) forces in a Lagrangian description of a system.

- a) Find the Lagrangian and equation of motion assuming only the spring force is acting with a spring constant $k = m\omega^2$.
- b) How could you modify the equation of motion directly to take into account air resistance?

For dissipating forces that are linear in velocity, such as the linear term for air resistance,

$$\mathbf{F}_{air} = -b\dot{\mathbf{r}},$$

we can write Lagrange's equations for particles that are both in a potential V that comes from conservative forces and that are *in addition* affected by dissipating forces linear in velocity, as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{K}}{\partial \dot{q}_j} = 0.$$

where $\mathcal{K}$ is the **dissipation function** given by:

$$\mathcal{K} = \sum_i \frac{1}{2} b_i \dot{\mathbf{r}}_i^2.$$

- c) Use the dissipation function to find the equation of motion for a mass attached to an ideal spring under the effect of air resistance from Lagrange's equation. Re-write the equation of motion in terms of the generalised momentum and the generalised coordinate.
- d) Find Hamilton's equations for this system. *Hint:* because of the dissipation of energy this is a non-Hamiltonian system, so *do not* try to find the Hamiltonian.
- e) Make a phase space plot for each of the three different cases:

$$i) \frac{b}{m} > 2\omega$$

$$ii) \frac{b}{m} = 2\omega$$

$$iii) \frac{b}{m} < 2\omega$$

Hint: The `matplotlib` function `streamplot` can be very useful here. This plots a vector flow from two coupled variables and their derivatives, which is exactly what we have in Hamilton's equations. An example routine `simple_phase_space_plot.py` can be found with this problem set.

- f) Show what we used above, namely that when adding a dissipating force,

$$\mathbf{F}_i^{\text{dis}} = -b_i \dot{\mathbf{r}}_i,$$

you can in general write Lagrange's equations as

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial \mathcal{K}}{\partial \dot{q}_j} = 0,$$

where $\mathcal{K}$ is the **dissipation function** given by:

$$\mathcal{K} = \sum_i \frac{1}{2} b_i \dot{\mathbf{r}}_i^2.$$

You may assume that any constraints are velocity independent.