

FYS3120 Classical Mechanics and
Electrodynamics

Problem set 4

Deadline for handing in (in canvas): **24 February**, 3pm

Problem 1 Symmetries and Noether's theorem in use

We start from the Lagrangian

$$L(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 - gm x, \quad (1)$$

which describes a free falling mass m with gravity in the negative x direction, or any other potential linear in the position.

We remind you that the criterion for the Lagrangian to give form symmetry for the resulting equations of motion under a transformation is

$$L[q'(q, t), \dot{q}'(q, \dot{q}, t), t'(q, t)] \frac{dt'}{dt} = L[q, \dot{q}, t] + \frac{dF(q, t)}{dt}. \quad (2)$$

- a) Show why the equation of motion from this Lagrangian is symmetric under time translations

$$t \rightarrow t' = t + a, \quad x \rightarrow x' = x,$$

and use Noether's theorem to find what quantity is conserved.

- b) Show why the equation of motion from this Lagrangian is symmetric under space translations

$$t \rightarrow t' = t, \quad x \rightarrow x' = x + a,$$

and use Noether's theorem to find what quantity is conserved. Find the physical meaning of the conserved quantity.

- c) Show that the **Galilean transformation** of the coordinates

$$t \rightarrow t' = t, \quad x \rightarrow x' = x - vt, \quad (3)$$

where v is a constant velocity, is a symmetry of this Lagrangian and will give the same equations of motion.

- d) Find what is conserved under the Galilean transformation.
- e) What is the most general possible Lagrangian with one generalised coordinate q and no explicit time-dependence, $L(q, \dot{q})$, that is invariant under Galilean transformations and gives distinct equations of motion? *Hint:* It could be wise to start from the criterion for symmetry in terms of infinitesimal transformations that was demonstrated in the lectures:

$$\sum_i \left[\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \left(\frac{d}{dt} \delta q_i - \dot{q}_i \frac{d}{dt} \delta t \right) \right] + \frac{\partial L}{\partial t} \delta t + L \frac{d}{dt} \delta t = \frac{d}{dt} \delta G. \quad (4)$$

- f) Find all the symmetries of the free-fall Lagrangian in (1) for infinitesimal transformations $\delta x(x, t)$ and $\delta t(x, t)$ of the coordinates. *Hint:* We are looking for the most general expressions that $\delta x(x, t)$ and $\delta t(x, t)$ can take. It can be useful to compare powers of $\dot{x}$ when writing out (4).

Problem 2 The brachistochrone problem

This is a classical problem in analytical mechanics. It was discussed by Galileo Galilei, who suggested a solution (but not the correct one), and studied the problem experimentally. In 1696 the problem was formulated as a challenge to the mathematicians at the time by Johann Bernoulli. He wrote in the journal *Acta Eruditorum*:¹

I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.

The problem he formulated was the following:

Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.

Five solutions were obtained from scientist and mathematicians we are all acquainted with, Newton, Jacob Bernoulli (the older brother of Johann), Leibniz and de L'Hôpital, in addition to Johann himself. Johann Bernoulli gave a formulation of the problem where he could use an analogy to Snell's law of refraction in optics to solve the problem.

Let us now rephrase the problem with a few more words: Assume a small body, named P in Fig. 1, moves in a vertical plane under the influence of gravity only. It leaves a point A with zero velocity and follows (without friction) a given path in the plane which passes through a second point B , as shown in the figure. Assume the path between the two points A and B can be changed, while the points themselves stay fixed. For which path

¹Johann Bernoulli, *Problema novum ad cujus solutionem Mathematici invitantur*, (A new problem to whose solution mathematicians are invited), *Acta Eruditorum* **18** (1696) 269.

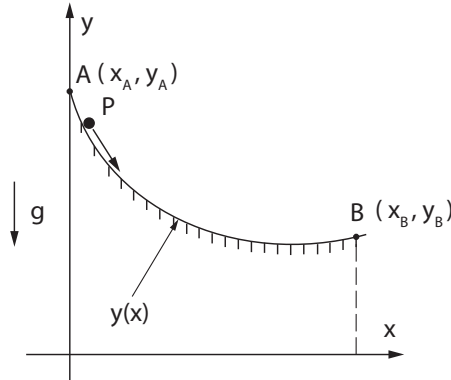


Figure 1: Illustration of the brachistochrone problem.

between the two points does the body spend the least time on the transit from point A to point B ?

The challenge for you is the following: Find the solution to the brachistochrone problem by using the correspondence between the variational problem (finding the “path of shortest time”) and the Lagrangian of mechanics, in the way discussed in the lectures.

The body P is to be treated as a point particle of mass m and the path is represented by a function $y(x)$ with x as the horizontal and y as the vertical coordinate. The boundary conditions, which fix the positions of point A and B , are specified as, $y(x_A) = y_A$, $y(x_B) = y_B$. To simplify the equations assume in the following that the initial coordinates are $x_A = y_A = 0$.

- a) Show that the time T spent by the body on the way between A and B can be expressed as an integral of the form

$$T[y(x)] = \int_{x_A}^{x_B} L(y, y') dx, \quad (5)$$

with $y' = \frac{dy}{dx}$, and with

$$L(y, y') = \sqrt{\frac{1 + y'^2}{-2gy}}. \quad (6)$$

Hint: Make use of energy conservation in the form $\frac{1}{2}mv^2 + mgy = 0$.

- b) In this case it is not so easy to find the equations of motion directly from Lagrange’s equation, however, we can use other properties from analytical mechanics. With $L(y, y')$ interpreted as a Lagrangian (x then plays the role of t in the usual formulation) the canonically conjugate momentum is $p = \frac{\partial L}{\partial y'}$ and the Hamiltonian is $H = py' - L$.

Explain why H is a constant of motion and use this fact to show that $y(x)$ satisfies a differential equation of the form

$$(1 + y'^2)y = -k^2, \quad (7)$$

with k as a constant.

- c) Equation (7) has a solution which can be written on *parametric form* as

$$x(\theta) = \frac{1}{2}k^2(\theta - \sin \theta), \quad (8)$$

$$y(\theta) = \frac{1}{2}k^2(\cos \theta - 1), \quad (9)$$

where θ has been introduced as a curve parameter. Show that (8) and (9) constitute a solution of the differential equation (7) by changing the variable from x to θ in (7), and by using the above expression for $y(\theta)$. In what way are the boundary conditions taken care of by this solution?