

FYS3120 Classical Mechanics and
Electrodynamics

Problem set 7

Deadline for handing in (in canvas): **7 April**, 3pm

Problem 1 Modified version of final-exam question in 2006

An electron, with charge e , moves in a constant electric field $\mathbf{E}$. The motion is determined by the relativistic Newton's equation

$$\frac{d}{dt}\mathbf{p} = e\mathbf{E}, \quad (1)$$

where $\mathbf{p}$ denotes the relativistic momentum $\mathbf{p} = m_e\gamma\mathbf{v}$, with m_e as the electron rest mass, $\mathbf{v}$ as the velocity and $\gamma = 1/\sqrt{1 - (v/c)^2}$ as the relativistic gamma factor. We assume the electron to move along the field lines, that is, there is no velocity component orthogonal to $\mathbf{E}$. *Hint:* We remind you that the relativistic energy can be written

$$E = \gamma m_e c^2 = \sqrt{p^2 c^2 + m_e^2 c^4}. \quad (2)$$

- a) Show that if $v = 0$ at time $t = 0$, then γ depends on time t as

$$\gamma = \sqrt{1 + \kappa^2 t^2}, \quad (3)$$

and determine κ .

- b) The proper time τ is related to the coordinate time t by the formula $\frac{dt}{d\tau} = \gamma$. Show that if we write γ using the rapidity η , $\gamma = \cosh \eta$, with the rapidity given by $\eta = \kappa\tau$, this satisfies the formula.
- c) For linear motion we have the following relation between the proper acceleration a_0 and the acceleration a measured in a fixed inertial reference frame, $a_0 = \gamma^3 a$. Use this to show that the electron has a constant proper acceleration, given by $\mathbf{a}_0 = e\mathbf{E}/m_e$. *Hint:* You will need to find the time-derivative of γ .
- d) Find explicit expressions for the four-velocity U^μ and the four-acceleration A^μ as functions of the proper time τ and show from these that we get the expected $A^\mu A_\mu = -a_0^2$. (For simplicity you may assume the motion to be in the x -direction.) As a reminder we also give the following functional relations:

$$\cosh^2 x - \sinh^2 x = 1, \quad \frac{d}{dx} \cosh x = \sinh x, \quad \frac{d}{dx} \sinh x = \cosh x. \quad (4)$$

Problem 2 Modified version of question from midterm exam 2006

A space ship leaves Earth orbit at local time $t = 0$ and heads for the closest star, *Proxima Centauri*, at a distance of $d = 4.2$ light years. The space ship starts with velocity $v = 0$ relative to the Earth, and during the first part of the journey it has a constant proper acceleration $a_0 = g$ in the direction of Proxima Centauri, with $g = 9.8 \text{ m/s}^2$ as the gravitational acceleration at the Earth surface.

At time τ_0 measured in the proper time of the space ship, the acceleration is reversed, so that $a_0 = -g$ and the velocity decreases until it reaches Proxima Centauri with final velocity $v = 0$. The first part of the trip, with acceleration $a_0 = g$, we refer to as part *I*, the second part, with $a_0 = -g$, as part *II*. The space ship visits Proxima Centauri only for a short time, and we neglect this time interval in our description of the journey. The return travel to Earth is carried out in the same way as the travel towards Proxima Centauri, and we refer to these parts of the journey as parts *III* and *IV*.

The coordinates of the Earth reference frame S are set to $x = 0$, $t = 0$ at departure of the space ship. Also the proper time τ is set to 0 at this event.

- a) Show that we can formulate a differential equation for the velocity of the space ship in terms of the rapidity η and the proper time τ as

$$\frac{d\eta}{d\tau} = \frac{g}{c}. \quad (5)$$

Hint: Use your knowledge of the relationship between the proper acceleration a_0 and the acceleration a in the Earth's frame to first find the differential $\frac{d\beta}{d\tau}$.

- b) Starting from a), show that for the first part of the journey (part I), the coordinates of the space ship in the Earth reference frame defines a space-time path,

$$x - x_I = \frac{c^2}{g} \cosh\left(\frac{g}{c}(\tau - \tau_I)\right), \quad t - t_I = \frac{c}{g} \sinh\left(\frac{g}{c}(\tau - \tau_I)\right), \quad (6)$$

where x_I , t_I and τ_I are constants. Similar expressions are valid for the other parts of the journey, but with other constants.

- c) Find the four-velocity and four-acceleration for Part I of the journey as a function of τ , and check that the proper acceleration implied by the path (6) is g . Explain why the space-time path is called hyperbolic.
- d) Determine the constants of Eq. (6) for Part I of the journey, and write the form of the equations with the correct constants. Sketch a two-dimensional Minkowski diagram which shows the full journey to Proxima Centauri and back.
- e) Determine the proper time value τ_0 . Find the total time of the journey as measured on Earth and on the space ship.
- f) What is the maximum speed reached by the space ship on the journey, as measured on Earth?