

FYS3120 Classical Mechanics and
Electrodynamics

Problem set 9

Deadline for handing in (in canvas): **21 April**, 3pm

Problem 1 Symmetries of relativistic Lagrangians

The idea here is to look at the consequences of Lorentz and translational invariance in a (relativistic) mechanical Lagrangian. Given a system with no constraints, where the action is given in terms of a Lorentz invariant Lagrangian parametrised by the proper time τ as

$$S = \int d\tau L(x^\mu(\tau), U^\mu(\tau)), \quad (1)$$

we will look at two transformations of the coordinates:

- The Lorentz transformations $x^\mu \rightarrow x'^\mu = L^\mu_\nu x^\nu$.
- The translation transformation $x^\mu \rightarrow x'^\mu = x^\mu + a^\mu$.

Together these are known as the Poincaré transformation.

- Explain why the Lorentz transformation is a symmetry of the system. You do not need to give a very formal argument.
- Assuming that the equations of motion are also invariant under translation, find the conserved Noether charge associated with the translation transformation. *Hint:* First show that we can write the Noether charge for our relativistic Lagrangian as

$$\delta Q = \frac{\partial L}{\partial U^\mu} \delta x^\mu - \frac{\partial L}{\partial U^\mu} U^\mu \delta \tau + L \delta \tau. \quad (2)$$

- Find the conserved Noether charge associated with the Lorentz transformation. Use that we can parameterise the Lorentz transformation by

$$L^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu, \quad (3)$$

where the parameters $\omega_{\mu\nu} = g_{\mu\rho} \omega^\rho_\nu$ form an antisymmetric tensor, $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

- For the free-particle Lagrangian

$$L = \frac{1}{2} m U^\mu U_\mu, \quad (4)$$

show that the conserved quantities under Lorentz transformations can be written as an antisymmetric tensor

$$L^{\mu\nu} = m(x^\mu U^\nu - U^\mu x^\nu), \quad (5)$$

and what is conserved under translation is

$$P^\mu = m U^\mu. \quad (6)$$

What physical properties do the components of $L^{\mu\nu}$ and P^μ represent?

Problem 2 We send a photon γ towards an electron e^- at rest.

- a) What is the minimum energy of the photon required for the following process to take place

$$\gamma + e^- \rightarrow e^- + e^- + e^+ ? \quad (7)$$

The particles e^- and e^+ have the same mass m_e .

- b) Show that the process

$$\gamma \rightarrow e^- + e^+, \quad (8)$$

is impossible.

Problem 3 A thin straight conducting cable lies at rest along the z -axis in an inertial reference frame S , and carries a constant current I . The cable is charge neutral in this frame.

- a) Show, by the use of Ampere's law, that the current produces a magnetic field $\mathbf{B} = B(r)\mathbf{e}_\phi$, where (r, ϕ) are polar coordinates in the (x, y) -plane, and $\mathbf{e}_\phi$ is a unit vector in the direction of increasing ϕ . Determine the function $B(r)$.

Consider next the same situation in a reference frame S' that moves with velocity v along the z -axis.

- b) Use the fact that charge and current densities transform under Lorentz transformation as components of a current four-vector to show that in S' the conducting cable will be charged. Determine the charge per unit length, λ' , and the current I' in this reference frame.
- c) Use Gauss' and Ampere's laws to determine the electric and magnetic fields, $\mathbf{E}'$ and $\mathbf{B}'$, as functions of the polar coordinates (r', ϕ') in reference frame S' .
- d) Show that if the fields in S' are derived from the fields in S by use of the relativistic transformation formulas for $\mathbf{E}$ and $\mathbf{B}$, this gives the same results as found in c).