General: *Each subproblem can give a maximal score of 5 points. Typically, 1 point is subtracted for minor calculational errors and inconsistent notation. The same may happen if a problem is solved in a much too complicated/messy way, clearly demonstrating lack of understanding of the basic techniques. No points are given for stating general formulas that are not relevant to the problem at hand. For each subproblem, the following solution contains a guide to how scores are to be assigned.*

Problem 1: Partial differential equation (PDE) and separation of variables

a) We are looking for solutions of the form $u(x,y) = F(x)G(y)$. Inserting into the PDE gives

$$\frac{G'}{c^2 G} = \frac{F''}{F} = k \tag{1}$$

where k is the separation constant. Now, k has to be negative for the following reason: If it were positive, $F(x)$ would be a linear combination of *real* exponential functions, $F(x) = A\exp\left(\sqrt{k}x\right) + B\exp\left(-\sqrt{k}x\right)$. But such a solution cannot simultaneously satisfy the BCs at $x = 0$ (which would imply $A = -B$) and at $x = a$, except for the trivial solution $A = B = 0$. Thus, we can only find a consistent solution for negative k, and we thus introduce the notation $k = -p^2$ where p is real. Thus, we find the ODEs

$$F'' + p^2 F = 0 \tag{2}$$
$$G' + (cp)^2 G = 0. \tag{3}$$

Grading:
* *Correct derivation of the ODEs (max 3pt),*
* *Correct argument for $k < 0$ (max 2 pt).*

b) Obviously, $F(x) = A\cos(px) + B\sin(px)$. Next, impose the BCs: $F(0) = 0$ implies $A = 0$. $F(a) = 0$ implies $pa = n\pi$. Thus,

$$F_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right). \tag{4}$$

Solving the equation for $G(x)$ gives $G(y) = \textit{constant} \cdot \exp\bigl(-(cp)^2 y\bigr)$. Introducing $\lambda_n \equiv cn\pi/a$, we thus have

$$G_n(y) = \tilde{B}_n e^{-\lambda_n^2 y}. \tag{5}$$

This gives the eigenfunctions

$$u_n(x,y) = C_n \sin\left(\frac{n\pi x}{a}\right) e^{-\lambda_n^2 y} \tag{6}$$

(In the above, B_n, $\tilde{B}_n$ and C_n are arbitrary constants of integration.)

Grading:
** Correct general solutions for F, G (max 2 pt).*
** Correct implementation of the BCs leading to p_n, λ_n or equivalent (max 2 pt).*
** Writing out $u_n(x, y)$ correctly (max 1 pt).*

c) The most general solution is a superposition of all the eigenfunctions,

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{a}\right) e^{-\lambda_n^2 y}. \tag{7}$$

This is a Fourier sine series with coefficients C_n determined by the remaining boundary condition, $u(x, 0) = u_0(x)$, in the usual way,

$$C_n = \frac{2}{a}\int_0^a u_0(x) \sin\left(\frac{n\pi x}{a}\right) dx \tag{8}$$

Grading:
** Up to 2.5 pt for each of these expressions.*
** They must be correct, including consistent notation (2's and a's in the right places, the expression for C_n must contain $u_0(x)$ explicitly, not replace e.g. a by L mid-problem etc)*

d) Inserting $u_0(x) = x$ into (8) we get

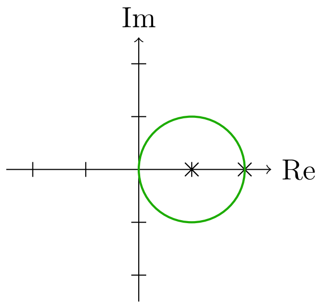
$$\begin{aligned} C_n &= \frac{2}{a}\int_0^a x \sin\left(\frac{n\pi x}{a}\right) dx & (9)\\ &= \frac{2}{a}\left\{-\left[\left(\frac{a}{n\pi}\right) x \cos\left(\frac{n\pi x}{a}\right)\right]_0^a + \int_0^a \left(\frac{a}{n\pi}\right) \cos\left(\frac{n\pi x}{a}\right) dx\right\} & (10)\\ &= -\frac{2a}{n\pi}\cos(n\pi) & (11)\\ &= \frac{2a}{n\pi}(-1)^{n+1} & (12) \end{aligned}$$

where we have used that the last integral in (10) is zero. This then gives the solution

$$u(x,y) = \frac{2a}{\pi}\left[\sin\left(\frac{\pi x}{a}\right) e^{-(c\pi/a)^2 y} - \frac{1}{2}\sin\left(\frac{2\pi x}{a}\right) e^{-4(c\pi/a)^2 y} + \frac{1}{3}\sin\left(\frac{3\pi x}{a}\right) e^{-9(c\pi/a)^2 y} - ...\right] \tag{13}$$

Grading:
** Correct expression for C_n (max 3 pt).*
** At least three terms of the solution written out in full detail (max 2 pt)*

Figure 1: Singularities and relevant disk of convergence for Problem 2a). We want the Laurent series to converge inside the disk, e.g. for $|z-1|<1$.



Problem 2: Laurent series

a) To find the singularities, we start by factorizing $f(z)$,

$$f(z) = \frac{1}{(z-1)(z-2)}, \tag{14}$$

so there are singularities at $z=1,2$. In order to find the residue at $z=1$ directly from the Laurent series, we have to Laurent expand around $z=1$, "near" $z=1$ to use Boas' formulation. There are two possible Laurent series around $z=1$: The one converging on the circular disk inside the nearest singularity (which is $z=2$), so for $|z-1|<1$, and the one converging for $|z-1|>1$. In other words, we have to find the Laurent series for $|z-1|<1$. The residue will then be the coefficient of the term $\sim \frac{1}{z-1}$. See figure.

Grading: *A full score answer should contain the following, giving 1 pt each:*
* *Correct factorization,*
* *Statement that we want expansion around* $z=1$*,*
* *Correct domain of convergence,*
* *Statement that Res = coefficient of* $1/(z-z_0)$ *term,*
* *Correct figure*

b) Given the answer in **a)**, we need an expansion of $f(z)$ *in powers of* $(z-1)$. The factor $1/(z-1)$ is already of this form and is kept as an overall factor. Then $1/(z-2)$ must be expanded in *positive* powers of $(z-1)$ to get the correct domain of convergence:

$$
\begin{aligned}
f(z) &= \frac{1}{z-1}\cdot\frac{1}{z-2} && (15)\\
&= \frac{1}{z-1}\cdot\frac{1}{(z-1)-1} && (16)\\
&= -\frac{1}{z-1}\cdot\frac{1}{1-(z-1)} && (17)\\
&= -\frac{1}{z-1}\sum_{n=0}^{\infty}(z-1)^n && (18)\\
&= -\frac{1}{z-1} - 1 - (z-1) - (z-1)^2 - ... && (19)
\end{aligned}
$$

The residue, i.e. the coefficient of the $1/(z-1)$ term is -1.

Grading:
** Correct Laurent series (max 4 pt).*
** Read off the correct coefficient to find the residue (max 1 pt).*

Problem 3: Series expansion solution for ODE

a) First of all, note that the form of the solution is prescribed by the problem text as a pure power series, so this is *not* supposed to be solved by the Frobenius method (although that would be possible). No indicial equation here. ;) Rather, let us insert the proposed power series into the DE term by term and display them in such a way that it is easy to read off and match coefficients power by power:

$$
\begin{aligned}
xy'' &= \sum_{n=0}^{\infty} n(n-1)\, c_n\, x^{n-1} && (20)\\
xy' &= \sum_{n=0}^{\infty} n\, c_n\, x^n && (21)\\
-y' &= \sum_{n=0}^{\infty} (-n)\, c_n\, x^{n-1} && (22)\\
-y &= \sum_{n=0}^{\infty} (-1)\, c_n\, x^n && (23)
\end{aligned}
$$

For each power of x, the sum of all the right hand sides must be zero. Let us collect terms for some given power x^k. This gives

$$
\begin{aligned}
(k+1)k\, c_{k+1} + k\, c_k - (k+1)\, c_{k+1} - c_k &= 0 && (24)\\
c_{k+1}\,[k(k+1)-(k+1)] + c_k\,(k-1) &= 0 && (25)\\
c_{k+1}\,(k+1)(k-1) + c_k\,(k-1) &= 0. && (26)
\end{aligned}
$$

As long as $k \neq 1$, we can divide by $k-1$ on both sides, which gives the desired result,

$$c_{k+1} = -\frac{c_k}{k+1}. \tag{27}$$

Now consider the case $k = 1$. From (26) we see that we get

$$c_2 \cdot 2 \cdot 0 + c_1 \cdot 0 = 0 \tag{28}$$

where we already know that $c_1 = -c_0 \neq 0$ since c_0 is non-zero by construction. This situation is very similar to an example we discussed in the lectures, and the interpretation is that c_2 is undetermined! This means we have already identified the two undetermined constants of this second order ODE: c_0 and c_2. Each of them will multiply one of the two linearly independent solutions.

Grading: *The following is required for full score:*
** Set up equations (20)-(23) [or equivalent] correctly (max 2 pt).*
** Derive the expression for c_k (max 2 pt).*
** Understand that c_2 is undetermined (max 1 pt)*

b) An easy way to see what happens is to study the first few terms explicitly: c_0 is undetermined and nonzero, and $c_1 = -c_0$. Then c_2 is undetermined, followed by

$$c_3 = -\frac{c_2}{3} = -2\frac{c_2}{3!} \tag{29}$$
$$c_4 = -\frac{c_3}{4} = 2\frac{c_2}{4!} \tag{30}$$

etc. Thus,

$$y(x) = c_0(1-x) + 2c_2\left(\frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - ...\right). \tag{31}$$

The part inside the parentheses is almost the series expansion of an exponential function, except that the constant and linear terms are missing:

$$y(x) = c_0(1-x) + 2c_2\left[e^{-x} - (1-x)\right] \tag{32}$$
$$= (c_0 - 2c_2)(1-x) + 2c_2 e^{-x}. \tag{33}$$

To simplify notation we can rename the constants, to conclude that the solution on closed form is

$$y(x) = A(1-x) + Be^{-x}. \tag{34}$$

Grading:
** Recognize series of* $\exp(-x)$ *(max 2 pt).*
** Organize into two linearly independent solutions with two undetermined constants (max 3 pt).*

Problem 4: Laplace transformations

a)

$$\mathcal{L}\{t \cdot H(t-1)\} = -\frac{d}{ds}\mathcal{L}\{H(t-1)\} \tag{35}$$
$$= -\frac{d}{ds}\left(\frac{e^{-s}}{s}\right) \tag{36}$$
$$= \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2} \tag{37}$$

Here we have used formulas (22) and (23) from the problem sheet (appendix), setting $f(t-a) = 1$ in (23) and recalling that $\mathcal{L}\{1\} = 1/s$, which is also found from (19) on the problem sheet if we set $n = 0$.

Grading: *There are several equivalent ways of solving this problem. Full score is given if the correct result is found either by this method or, for example, convolution.*

b) Let us do the first calculation by convolution:

$$f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} \tag{38}$$
$$= \mathcal{L}^{-1}\{\mathcal{L}\{1\} \cdot \mathcal{L}\{\sin t\}\} \tag{39}$$
$$= 1 \star \sin t \tag{40}$$
$$= \int_0^t \sin \tau d\tau \tag{41}$$
$$= 1 - \cos t. \tag{42}$$

Here we have used (19) (with $n = 0$) and (20) (with $a = 1$) from the problem sheet (appendix). The second sub-problem can again be solved by convolution, noting that

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s} \cdot \frac{1}{s(s^2+1)}, \tag{43}$$

where the inverse Laplace transform of the last factor was just found to be $1 - \cos t$. So the inverse Laplace transform of (43) can be found as $1 \star (1 - \cos t) = \int_0^t (1 - \cos \tau) d\tau$, which is a straightforward integral. Alternatively, we can use partial fraction decomposition to split up the expression, then read off the inverse directly, using tabulated formulas:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{(s^2+1)}\right\} = t - \sin t. \tag{44}$$

Grading: *There are several equivalent ways of solving this problem. Full score is given if the correct result is found by any valid method. Max score is* **2.5 pt** *for each of the two calculations.*

c) For the proof, differentiate the definition of Laplace transform with respect to s,

$$\begin{aligned} F'(s) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)dt = \int_0^\infty e^{-st}(-t)f(t)dt && (45)\\ &= -\mathcal{L}\{t\cdot f(t)\} \quad q.e.d. && (46)\end{aligned}$$

To use this for the problem at hand, define

$$\frac{s}{(s^2+k^2)^2} \equiv -F'(s). \tag{47}$$

The antiderivative is easy, giving

$$F(s) = \frac{1}{2}\frac{1}{s^2+k^2} = \frac{1}{2k}\frac{k}{s^2+k^2}. \tag{48}$$

Thus

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+k^2)^2}\right\} = \frac{t}{2k}\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \frac{t}{2k}\sin(kt). \tag{49}$$

Grading:
** Correct proof (max 1 pt).*
** Max 4 pt for getting the second (main) part right.*

d) An important thing to realize in this problem is that many of the calculations needed have already been done in **a)-c)**, so double work should be avoided.

We will follow the usual procedure: **(i)** Laplace transform the entire differential equation, which automatically implements the boundary conditions. **(ii)** Solve algebraically for $Y(s)$, i.e. the Laplace transform of the solution $y(t)$. **(iii)** Inverse Laplace transform of $Y(s)$ to find the solution $y(t)$. We start from the DE

$$y'' + y = r(t) \tag{50}$$

so that its Laplace transform, with $y(0) = y'(0)$, is

$$(s^2+1)Y(s) = \mathcal{L}\{r(t)\}. \tag{51}$$

To find $\mathcal{L}\{r(t)\}$, first note that $r(t)$ can be expressed as

$$r(t) = t[1-H(t-1)] = t - t\cdot H(t-1). \tag{52}$$

Thus we can immediately write down, without further calculations,

$$\mathcal{L}\{r(t)\} = \frac{1}{s^2} - \frac{e^{-s}}{s} - \frac{e^{-s}}{s^2}, \tag{53}$$

where the first term is found from the table, and the last two terms were calculated in **a)**. **Alternatively**, if one does not feel comfortable manouvering with step functions, $\mathcal{L}\{r(t)\}$, can be found directly,

$$\mathcal{L}\{r(t)\} = \int_0^\infty e^{-st} f(t) dt = \int_0^1 t e^{-st} dt, \tag{54}$$

which, after a quick integration by parts, gives the same result. Either way one must Laplace transform $r(t)$ in order to properly take into account the two regimes $t < 1$ and $t > 1$. Many students took a shortcut here, giving the wrong result.

Combining (51) and (53) and solving for $Y(s)$ gives

$$Y(s) = \frac{1}{s^2(s^2+1)} - \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)}. \tag{55}$$

The final task is to find the inverse Laplace transform of this expression. Again, no calculations needed if we use the results of **b)** along with t-shifting (eq.(23) from the problem sheet):

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = t - \sin t \tag{56}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s^2(s^2+1)}\right\} = [(t-1) - \sin(t-1)]\; H(t-1) \tag{57}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = [1 - \cos(t-1)]\; H(t-1) \tag{58}$$

Thus, the solution of our DE is

$$y(t) = t - \sin t - H(t-1)\,[t - \sin(t-1) - \cos(t-1)]\,. \tag{59}$$

Written out explicitly,

$$y(t) = \begin{cases} t - \sin t, \quad 0 \le t < 1 \\ -\sin t + \sin(t-1) + \cos(t-1), \quad t > 1 \end{cases} \tag{60}$$

Grading:
** The student must show that (s)he understands the general approach for solving DEs by Laplace transform, i.e. (i) transform the DE, (ii) solve for $Y(s)$, and (iii) transform back to find $y(t)$ (max 1 pt).*
** Get the calculation right, up until (59) (max 3 pt).*
** Finally, write out the solution explicitly as in (60) (max 1 pt).*
* **No points will be given for trying to solve the differential equation by other methods.**