Formula sheet

Linear algebra

Cauchy–Schwarz:

$$|\langle\phi|\psi\rangle|^2 \leq \| \, |\phi\rangle \|^2 \, \| \, |\psi\rangle \|^2$$

Triangle inequality:

$$\| \, |\psi\rangle + |\phi\rangle \| \leq \| \, |\psi\rangle \| + \| \, |\phi\rangle \|$$

Completeness:

$$I = \sum_i |i\rangle\langle i|$$

Trace:

$$\mathrm{tr}(A) = \sum_i \langle i| A \, |i\rangle$$

Partial trace:

$$\mathrm{tr}_B(\rho_{AB}) = \sum_i {}_B\langle i| \, \rho_{AB} \, |i\rangle_B$$

Spectral decomposition:

$$A = \sum_k \lambda_k \, |v_k\rangle\langle v_k|$$

Operator functions:

$$f(A) = \sum_k f(\lambda_k) \, |v_k\rangle\langle v_k|$$

Absolute value:

$$|A| = \sqrt{A^\dagger A}$$

Normal operator:

$$A^\dagger A = A A^\dagger$$

Hermitian operator:

$$H = H^\dagger$$

Unitary operator:

$$U^\dagger U = U U^\dagger = I$$

Positive operator:

$$E \geq 0 \iff \langle\psi| E \, |\psi\rangle \geq 0 \quad \forall \, |\psi\rangle$$

Projector (orthogonal):

$$P^2 = P, \quad P^\dagger = P$$

States

Z-basis (computational basis) states:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

X-basis states:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Y-basis states:

$$|+i\rangle = \frac{|0\rangle + i \, |1\rangle}{\sqrt{2}}, \quad |-i\rangle = \frac{|0\rangle - i \, |1\rangle}{\sqrt{2}}$$

Qubit pure state:

$$|\psi\rangle = \alpha \, |0\rangle + \beta \, |1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1$$

Bloch vector:

$$\begin{aligned} \vec{r} &= (\sin\theta\cos\phi, \, \sin\theta\sin\phi, \, \cos\theta) \\ &= \langle\psi| \, \vec{\sigma} \, |\psi\rangle, \quad \vec{\sigma} = (X, Y, Z) \end{aligned}$$

Bell states:

$$\left|\Psi^+\right\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\Psi^-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$
$$\left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\Phi^-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

Schmidt decomposition:

$$|\psi\rangle_{AB} = \sum_k \sqrt{\lambda_k} \, |u_k\rangle_A \, |v_k\rangle_B,$$
$$\lambda_k \geq 0, \quad \sum_k \lambda_k = 1$$

Density operators

Density operator:

$$\rho = \rho^\dagger, \quad \rho \geq 0, \quad \operatorname{tr}(\rho) = 1$$

Ensemble:

$$\rho = \sum_k p_k \left|\psi_k\right\rangle\!\left\langle\psi_k\right|,$$
$$p_k \geq 0, \quad \sum_k p_k = 1$$

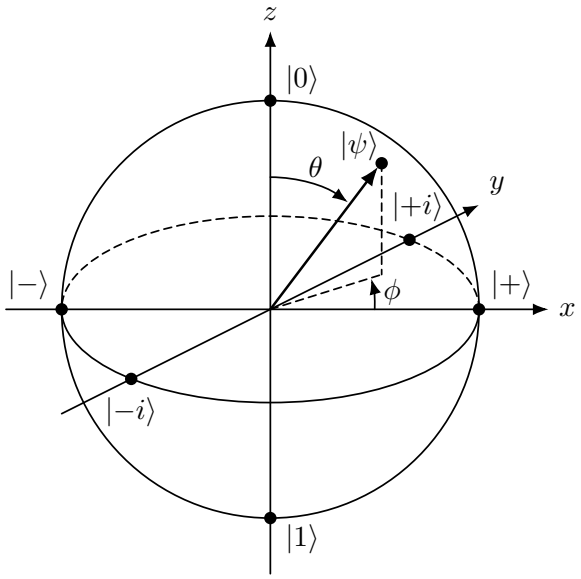
Bloch vector:

$$\vec{r} = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)$$
$$= \operatorname{tr}(\rho\,\vec{\sigma}), \quad \vec{\sigma} = (X, Y, Z)$$

Purification:

$$\rho_A = \sum_k \lambda_k \left|v_k\right\rangle\!\left\langle v_k\right|$$
$$\implies \left|\psi\right\rangle_{AR} = \sum_k \sqrt{\lambda_k}\left|v_k\right\rangle_A \left|k\right\rangle_R$$

Bloch sphere



Sine and cosine values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

Single-qubit gates

Matrix representations are in the computational basis.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{NOT} = X = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$Z = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$S = \sqrt{Z} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$T = \sqrt{S} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

$$P(\varphi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

$$U_{\hat{n}}(\varphi) = e^{-i\frac{\varphi}{2}(\hat{n}\cdot\vec{\sigma})} = \cos\frac{\varphi}{2}\,I - i\sin\frac{\varphi}{2}\hat{n}\cdot\vec{\sigma}$$

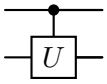
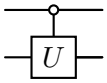
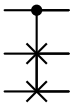
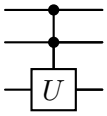
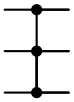
$$U_x(\varphi) = e^{-i\frac{\varphi}{2}X} = \cos\frac{\varphi}{2}\,I - i\sin\frac{\varphi}{2}\,X$$
$$= \begin{pmatrix} \cos\frac{\varphi}{2} & -i\sin\frac{\varphi}{2} \\ -i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}$$

$$U_y(\varphi) = e^{-i\frac{\varphi}{2}Y} = \cos\frac{\varphi}{2}\,I - i\sin\frac{\varphi}{2}\,Y$$
$$= \begin{pmatrix} \cos\frac{\varphi}{2} & -\sin\frac{\varphi}{2} \\ \sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}$$

$$U_z(\varphi) = e^{-i\frac{\varphi}{2}Z} = \cos\frac{\varphi}{2}\,I - i\sin\frac{\varphi}{2}\,Z$$
$$= \begin{pmatrix} e^{-i\frac{\varphi}{2}} & 0 \\ 0 & e^{i\frac{\varphi}{2}} \end{pmatrix}$$

Multi-qubit gates

In the expressions below, tensor factors from left to right correspond to wires from top to bottom.

Gate	Circuit	Expression
CU		$\|0\rangle\langle 0\| \otimes I_2 + \|1\rangle\langle 1\| \otimes U$
Anti-CU		$\|0\rangle\langle 0\| \otimes U + \|1\rangle\langle 1\| \otimes I_2$
CNOT = CX		$\|0\rangle\langle 0\| \otimes I_2 + \|1\rangle\langle 1\| \otimes X$
CZ		$\|0\rangle\langle 0\| \otimes I_2 + \|1\rangle\langle 1\| \otimes Z$
SWAP		$\sum_{x,y\in\{0,1\}} \|x\,y\rangle\langle y\,x\|$
Fredkin = CSWAP		$\|0\rangle\langle 0\| \otimes I_4 + \|1\rangle\langle 1\| \otimes \mathrm{SWAP}$
CCU		$(I_4 - \|11\rangle\langle 11\|) \otimes I_2 + \|11\rangle\langle 11\| \otimes U$
Toffoli = CCNOT = CCX		$(I_4 - \|11\rangle\langle 11\|) \otimes I_2 + \|11\rangle\langle 11\| \otimes X$
CCZ		$(I_4 - \|11\rangle\langle 11\|) \otimes I_2 + \|11\rangle\langle 11\| \otimes Z$

Pauli algebra

$$\sigma_i\sigma_j = \delta_{ij}I + i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k$$

$$[\sigma_i, \sigma_j] = 2i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}I$$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} +1, & ijk \text{ even perm. of } 123 \\ -1, & ijk \text{ odd perm. of } 123 \\ 0, & \text{otherwise} \end{cases}$$

Quantum channels

Definition:

$$\mathcal{E}(\rho) = \mathrm{tr}_E\Big[U(\rho \otimes |0\rangle\langle 0|_E)U^\dagger\Big]$$

Operator-sum form:

$$\mathcal{E}(\rho) = \sum_k M_k \rho M_k^\dagger, \quad \sum_k M_k^\dagger M_k = I$$

Kraus operators / operation elements:

$$M_k = {}_E\langle k|\, U\, |0\rangle_E$$

Projective measurements: $\{P_m\}$

Definition:

$$\sum_m P_m = I,$$
$$P_m = P_m^\dagger,$$
$$P_m P_n = \delta_{mn} P_m$$

Probability:

$$p(m) = \mathrm{tr}(P_m \rho)$$

State update:

$$\rho \mapsto \frac{P_m \rho P_m}{\mathrm{tr}(P_m \rho)}$$

Observable:

$$A = \sum_a a P_a$$

Expectation value:

$$\langle A \rangle_\rho = \mathrm{tr}(A\rho)$$

General measurements: $\{M_m\}$

Definition:

$$\sum_m M_m^\dagger M_m = I$$

Probability:

$$p(m) = \mathrm{tr}\Big(M_m \rho M_m^\dagger\Big)$$

State update:

$$\rho \mapsto \frac{M_m \rho M_m^\dagger}{\mathrm{tr}\Big(M_m \rho M_m^\dagger\Big)}$$

POVM measurements: $\{E_m\}$

Definition:

$$\sum_m E_m = I,$$
$$E_m \geq 0$$

Probability:

$$p(m) = \mathrm{tr}(E_m \rho)$$

State discrimination

Trace distance:

$$D(\rho, \sigma) = \frac{1}{2} \mathrm{tr}\,|\rho - \sigma|$$

Trace distance for pure states:

$$D(|\psi\rangle, |\phi\rangle) = \sqrt{1 - |\langle\psi|\phi\rangle|^2}$$

Trace distance operational:

$$D(\rho, \sigma) = \max_{0 \leq E \leq I} |\mathrm{tr}(E\rho) - \mathrm{tr}(E\sigma)|$$

Triangle inequality:

$$D(\rho, \tau) \leq D(\rho, \sigma) + D(\sigma, \tau)$$

Joint convexity of trace distance:

$$D\left(\sum_x p_x \rho_x, \sum_x p_x \sigma_x\right) \leq \sum_x p_x D(\rho_x, \sigma_x)$$

Fidelity:

$$F(\rho, \sigma) = \mathrm{tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}$$

Fidelity for pure states:

$$F(|\psi\rangle, |\phi\rangle) = |\langle\psi|\phi\rangle|$$

Fidelity with one pure state:

$$F(|\psi\rangle, \sigma) = \sqrt{\langle\psi|\,\sigma\,|\psi\rangle}$$

Uhlmann's theorem:

$$F(\rho_A, \sigma_A) = \max_{\substack{|\psi_\rho\rangle_{AR}, |\phi_\sigma\rangle_{AR} \\ \text{purifications}}} |\langle\psi_\rho|\phi_\sigma\rangle|$$

Joint concavity of fidelity:

$$F\left(\sum_x p_x \rho_x, \sum_x p_x \sigma_x\right) \geq \sum_x p_x F(\rho_x, \sigma_x)$$

Relations:

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \leq \sqrt{1 - F(\rho, \sigma)^2}$$

$$D(|\psi\rangle, |\phi\rangle) = \sqrt{1 - F(|\psi\rangle, |\phi\rangle)^2}$$

Error correction

Probability of getting exactly n errors among N independent trials with error probability p:

$$\Pr(n) = \binom{N}{n} p^n (1-p)^{N-n},$$

$$\binom{N}{n} = \frac{N!}{n!\,(N-n)!}$$

Bit-flip code encoding:

$$|0_\mathrm{L}\rangle = |000\rangle, \quad |1_\mathrm{L}\rangle = |111\rangle$$

Phase-flip code encoding:

$$|0_\mathrm{L}\rangle = |+++\rangle, \quad |1_\mathrm{L}\rangle = |---\rangle$$

Shor code encoding:

$$|0_\mathrm{L}\rangle = \frac{(|000\rangle + |111\rangle)^{\otimes 3}}{2\sqrt{2}},$$

$$|1_\mathrm{L}\rangle = \frac{(|000\rangle - |111\rangle)^{\otimes 3}}{2\sqrt{2}}$$

Classical entropy

Convention: $0 \log 0 = 0$.

Binary entropy:

$$h_\mathrm{b}(p) = -p \log_2 p - (1-p) \log_2 (1-p)$$

Shannon entropy:

$$H(X) = -\sum_x p(x) \log_2 p(x)$$

Joint entropy:

$$H(X,Y) = -\sum_{x,y} p(x,y) \log_2 p(x,y)$$

Conditional entropy:

$$\begin{aligned} H(X|Y) &= -\sum_{x,y} p(x,y) \log_2 p(x|y) \\ &= H(X,Y) - H(Y) \end{aligned}$$

Mutual information:

$$I(X:Y) = H(X) + H(Y) - H(X,Y)$$

Relative entropy:

$$H(p\|q) = \sum_x p(x) \log_2 \frac{p(x)}{q(x)}$$

Quantum entropy

Von Neumann entropy:

$$S(\rho) = -\operatorname{tr}(\rho \log_2 \rho)$$

Joint entropy:

$$S(A,B) = S(\rho_{AB})$$

Conditional entropy:

$$S(A|B) = S(A,B) - S(B)$$

Mutual information:

$$I(A:B) = S(A) + S(B) - S(A,B)$$

Relative entropy:

$$S(\rho\|\sigma) = \operatorname{tr}(\rho \log_2 \rho) - \operatorname{tr}(\rho \log_2 \sigma)$$

Holevo quantity:

$$\chi = S\left(\sum_x p_x \rho_x\right) - \sum_x p_x S(\rho_x)$$

Holevo bound:

$$I(X:Y) \leq \chi$$

Bell tests

Correlator:

$$E(x,y) = \sum_{a,b=\pm 1} ab\, p(a,b|x,y)$$

CHSH parameter:

$$S = E(0,0) + E(0,1) + E(1,0) - E(1,1)$$

Local hidden-variable theory:

$$|S| \leq 2$$