University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 0

This problem set will be discussed in the group exercise session on **Jan. 21**.

Problem 1: Bras, inner products, outer products, and adjoints in $\mathbb{C}^3$ (J. Gulla, O. Leinonen)

In this exercise we work in $\mathbb{C}^3$ with a fixed orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Let

$$|\psi\rangle = \begin{pmatrix} 1+i \\ i \\ 0 \end{pmatrix} \qquad \text{and} \qquad |\phi\rangle = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}. \tag{1}$$

a) Write down the bras $\langle\psi|$ and $\langle\phi|$.

b) Compute $\langle\psi|\phi\rangle$ and decide whether $|\psi\rangle$ and $|\phi\rangle$ are orthogonal. Verify conjugate symmetry $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$ for these vectors.

c) Compute $\| |\psi\rangle\|$ and $\| |\phi\rangle\|$, and write down normalized vectors $|\psi_{\mathrm{N}}\rangle$ and $|\phi_{\mathrm{N}}\rangle$.

d) Compute the outer product $A = |\psi\rangle\langle\phi|$ and write it as a 3×3 matrix in the basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Show that for any vector $|v\rangle$ one has

$$A\,|v\rangle = |\psi\rangle\,\langle\phi|v\rangle. \tag{2}$$

Then compute $A\,|\phi\rangle$ and $A\,|\psi\rangle$.

e) Compute the adjoint $A^\dagger$ in bra–ket form and as a 3×3 matrix. Briefly interpret what taking the adjoint does for a rank-one operator.

Problem 2: Orthogonality, normalization, bases, and transformations (J. Gulla, O. Leinonen)

Throughout this exercise we work in the computational basis $\{|0\rangle, |1\rangle\}$. Consider the two (unnormalized) vectors

$$|v\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad |w\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{3}$$

a) Show that $|v\rangle$ and $|w\rangle$ are orthogonal. Find the normalized vectors $|v_{\mathrm{N}}\rangle$ and $|w_{\mathrm{N}}\rangle$.

b) Show that $\{|v_{\mathrm{N}}\rangle, |w_{\mathrm{N}}\rangle\}$ is an orthonormal basis. Verify the completeness relation (resolution of the identity)

$$|v_{\mathrm{N}}\rangle\langle v_{\mathrm{N}}| + |w_{\mathrm{N}}\rangle\langle w_{\mathrm{N}}| = \mathbb{1}. \tag{4}$$

c) Use (4) to express $|0\rangle$ and $|1\rangle$ as linear combinations of $|v_{\mathrm{N}}\rangle$ and $|w_{\mathrm{N}}\rangle$. Next, define bases $\mathcal{B} = \{|0\rangle, |1\rangle\}$ and $\mathcal{B}' = \{|v_{\mathrm{N}}\rangle, |w_{\mathrm{N}}\rangle\}$. For any state $|\psi\rangle$, define coordinate column vectors $[\psi]_{\mathcal{B}}$ and $[\psi]_{\mathcal{B}'}$ by

$$|\psi\rangle = \big([\psi]_{\mathcal{B}}\big)_0\,|0\rangle + \big([\psi]_{\mathcal{B}}\big)_1\,|1\rangle = \big([\psi]_{\mathcal{B}'}\big)_0\,|v_{\mathrm{N}}\rangle + \big([\psi]_{\mathcal{B}'}\big)_1\,|w_{\mathrm{N}}\rangle. \tag{5}$$

The *change-of-basis matrix* U from $\mathcal{B}'$ to $\mathcal{B}$ is defined by

$$[\psi]_{\mathcal{B}} = U\,[\psi]_{\mathcal{B}'} \qquad \text{for all } |\psi\rangle. \tag{6}$$

Find U. Hint[1].

d) Show that the change-of-basis matrix U you found is unitary. Interpret $U^\dagger$ in terms of changing coordinates between $\mathcal{B}$ and $\mathcal{B}'$.

e) The relation $[\psi]_{\mathcal{B}} = U[\psi]_{\mathcal{B}'}$ describes a *passive* change of coordinates (same state, new basis). In contrast, an *active* unitary maps the state to a new state $|\psi\rangle \mapsto U\,|\psi\rangle$, whose coordinates in the *same* basis satisfy $[U\,|\psi\rangle]_{\mathcal{B}} = U[\psi]_{\mathcal{B}}$. Explain in 1–2 sentences what is held fixed in each case.

Problem 3: Projectors and changing basis for operators (J. Gulla)

Unless otherwise stated, we work in the computational basis $\{|0\rangle, |1\rangle\}$. Let

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{7}$$

Note: $\{|+\rangle, |-\rangle\}$ is the same basis as $\{|v_{\mathrm{N}}\rangle, |w_{\mathrm{N}}\rangle\}$ from Problem 2, so the same Hadamard matrix H will reappear below.

a) Compute the projector $P = |+\rangle\langle+|$ as a 2×2 matrix in the computational basis. Verify that $P^2 = P$ and that $P^\dagger = P$.

b) Compute $P\,|+\rangle$ and $P\,|-\rangle$ and interpret the result geometrically. Conclude what the eigenvalues of P are.

c) Let $\mathcal{B}_\pm = \{|+\rangle, |-\rangle\}$. Compute the matrix representation $[P]_{\mathcal{B}_\pm}$ of P in the basis $\mathcal{B}_\pm$. Compare it with the matrix representation of P in the computational basis. Hint[2].

d) Let U be the change-of-basis matrix from $\mathcal{B}_\pm$ to the computational basis. (This is exactly the matrix $U = H$ found in Problem 2.) Since we found that U is unitary, verify that

$$[P]_{\mathcal{B}_\pm} = U^\dagger\,[P]_{\{|0\rangle,|1\rangle\}}\,U. \tag{8}$$

(*General rule:* for any operator A, one has $[A]_{\mathcal{B}'} = S^{-1}[A]_{\mathcal{B}}S$ for a change-of-basis matrix S.)

e) Let $Q = \mathbb{1} - P$. Show that $Q = |-\rangle\langle-|$ and that $PQ = 0$, and verify the completeness relation

$$P + Q = \mathbb{1}. \tag{9}$$

Problem 4: Pauli matrices (Adapted from Mike & Ike 2.11)

The Pauli matrices are defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{10}$$

Throughout this exercise we work in the computational basis $\{|0\rangle, |1\rangle\}$.

[1] For the change-of-basis matrix U, compute where it sends the coordinate vectors $[v_{\mathrm{N}}]_{\mathcal{B}'}$ and $[w_{\mathrm{N}}]_{\mathcal{B}'}$.
[2] Compute matrix elements $([P]_{\mathcal{B}_\pm})_{ij} = \langle b_i|P\,|b_j\rangle$ where $\mathcal{B}_\pm = \{|b_0\rangle, |b_1\rangle\} = \{|+\rangle, |-\rangle\}$.

a) Write each Pauli matrix as a sum of outer products in the computational basis, i.e.

$$\sigma = \sum_{i,j\in\{0,1\}} \big(\langle i|\sigma\,|j\rangle\big)\ |i\rangle\langle j|. \tag{11}$$

b) Find the eigenvalues and a set of normalized eigenvectors for each of $\sigma_x, \sigma_y, \sigma_z$. Hint[3].

c) Use your eigenpairs to write the spectral decomposition

$$\sigma = \sum_k \lambda_k\ |v_k\rangle\langle v_k|, \tag{12}$$

for each Pauli matrix σ. Explain briefly the difference between (11) and (12).

d) State what the matrix representation of each $\sigma_x, \sigma_y, \sigma_z$ looks like in its own eigenbasis.

e) Use the spectral decomposition to express the projectors onto the ± 1 eigenspaces of each Pauli matrix σ as simple linear combinations of $\mathbb{1}$ and σ. Verify that these two projectors sum to the identity.

Problem 5: Unitary matrices (J. Gulla)

a) Let U be unitary. Show that unitaries preserve inner products:

$$\langle U\psi|U\phi\rangle = \langle\psi|\phi\rangle \qquad \text{for all } |\psi\rangle, |\phi\rangle. \tag{13}$$

Conclude that unitaries preserve norms.

b) Show that all eigenvalues of a unitary matrix have modulus 1, i.e. can be written as $e^{i\theta}$ for some real θ. Hint[4].

Problem 6: Commutators (J. Bergli)

We use the standard notation for commutators and anticommutators

$$[A, B] = AB - BA, \qquad \{A, B\} = AB + BA, \tag{14}$$

where A and B are operators or matrices. Show the following identities.

a) Show that

$$[A, BC] = [A, B]\,C + B\,[A, C]. \tag{15}$$

b) Prove that $(AB)^\dagger = B^\dagger A^\dagger$. Use this to show that

$$[A, B]^\dagger = [B^\dagger, A^\dagger], \qquad \{A, B\}^\dagger = \{B^\dagger, A^\dagger\}. \tag{16}$$

[3] For some matrix σ, use the standard eigenvalue method: solve the characteristic equation $\det(\sigma - \lambda\mathbb{1}) = 0$ to find the eigenvalues λ. For each eigenvalue, solve $(\sigma - \lambda\mathbb{1})\,|v\rangle = \mathbf{0}$ to find an eigenvector $|v\rangle$, and then normalize it.

[4] If $U\,|v\rangle = \lambda\,|v\rangle$, compare $\|U\,|v\rangle\|$ and $\|\lambda\,|v\rangle\|$.