University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 0

This problem set will be discussed in the group exercise session on **Jan. 21**.

Problem 1: Bras, inner products, outer products, and adjoints in $\mathbb{C}^3$ (J. Gulla, O. Leinonen)

In this exercise we work in $\mathbb{C}^3$ with a fixed orthonormal basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Let

$$|\psi\rangle = \begin{pmatrix} 1+i \\ i \\ 0 \end{pmatrix} \qquad \text{and} \qquad |\phi\rangle = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}. \tag{1}$$

a) Write down the bras $\langle\psi|$ and $\langle\phi|$.

Solution: The bra is the Hermitian conjugate (transpose + complex conjugate). Hence

$$\langle\psi| = \begin{pmatrix} 1-i & -i & 0 \end{pmatrix}, \qquad \langle\phi| = \begin{pmatrix} 0 & 1 & -i \end{pmatrix}. \tag{2}$$

b) Compute $\langle\psi|\phi\rangle$ and decide whether $|\psi\rangle$ and $|\phi\rangle$ are orthogonal. Verify conjugate symmetry $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$ for these vectors.

Solution:

$$\langle\psi|\phi\rangle = \begin{pmatrix} 1-i & -i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = (1-i)\cdot 0 + (-i)\cdot 1 + 0 \cdot i = -i. \tag{3}$$

Since $\langle\psi|\phi\rangle \neq 0$, the vectors are not orthogonal.

Next,

$$\langle\psi|\phi\rangle^* = (-i)^* = i, \tag{4}$$

and directly

$$\langle\phi|\psi\rangle = \begin{pmatrix} 0 & 1 & -i \end{pmatrix} \begin{pmatrix} 1+i \\ i \\ 0 \end{pmatrix} = 0 \cdot (1+i) + 1 \cdot i + (-i) \cdot 0 = i. \tag{5}$$

Thus $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$.

c) Compute $\| |\psi\rangle \|$ and $\| |\phi\rangle \|$, and write down normalized vectors $|\psi_{\mathrm{N}}\rangle$ and $|\phi_{\mathrm{N}}\rangle$.

Solution:

$$\| |\psi\rangle \|^2 = \langle\psi|\psi\rangle = |1+i|^2 + |i|^2 + |0|^2 = 2 + 1 + 0 = 3, \tag{6}$$

$$\| |\phi\rangle \|^2 = \langle\phi|\phi\rangle = |0|^2 + |1|^2 + |i|^2 = 0 + 1 + 1 = 2. \tag{7}$$

Therefore

$$|\psi_{\mathrm{N}}\rangle = \frac{1}{\sqrt{3}} \begin{pmatrix} 1+i \\ i \\ 0 \end{pmatrix}, \qquad |\phi_{\mathrm{N}}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}. \tag{8}$$

d) Compute the outer product $A = |\psi\rangle\langle\phi|$ and write it as a 3×3 matrix in the basis $\{|0\rangle, |1\rangle, |2\rangle\}$. Show that for any vector $|v\rangle$ one has

$$A\,|v\rangle = |\psi\rangle\,\langle\phi|v\rangle. \tag{9}$$

Then compute $A\,|\phi\rangle$ and $A\,|\psi\rangle$.

Solution: First,

$$A = |\psi\rangle\langle\phi| = \begin{pmatrix} 1+i \\ i \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & -i \end{pmatrix} = \begin{pmatrix} 0 & 1+i & 1-i \\ 0 & i & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{10}$$

For (9), note that $\langle\phi|\,|v\rangle$ is a scalar. By associativity of matrix multiplication,

$$(|\psi\rangle\langle\phi|)\,|v\rangle = |\psi\rangle\,(\langle\phi|\,|v\rangle) = |\psi\rangle\,\langle\phi|v\rangle. \tag{11}$$

This shows A is a rank-one operator: it takes the overlap with $|\phi\rangle$ and outputs a multiple of $|\psi\rangle$. Applying it to $|\phi\rangle$ gives

$$A\,|\phi\rangle = |\psi\rangle\langle\phi|\phi\rangle = 2\,|\psi\rangle. \tag{12}$$

Applying it to $|\psi\rangle$ gives

$$A\,|\psi\rangle = |\psi\rangle\langle\phi|\psi\rangle = |\psi\rangle\,(i) = i\;|\psi\rangle. \tag{13}$$

We can also compute these actions directly by matrix multiplication using the 3×3 matrix for A:

$$A\,|\phi\rangle = \begin{pmatrix} 0 & 1+i & 1-i \\ 0 & i & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix} = \begin{pmatrix} 2+2i \\ 2i \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1+i \\ i \\ 0 \end{pmatrix} = 2\,|\psi\rangle, \tag{14}$$

$$A\,|\psi\rangle = \begin{pmatrix} 0 & 1+i & 1-i \\ 0 & i & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1+i \\ i \\ 0 \end{pmatrix} = \begin{pmatrix} -1+i \\ -1 \\ 0 \end{pmatrix} = i \begin{pmatrix} 1+i \\ i \\ 0 \end{pmatrix} = i\,|\psi\rangle. \tag{15}$$

e) Compute the adjoint $A^\dagger$ in bra–ket form and as a 3×3 matrix. Briefly interpret what taking the adjoint does for a rank-one operator.

Solution: Using $(|v\rangle\langle w|)^\dagger = |w\rangle\langle v|$, we get

$$A^\dagger = (|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi|. \tag{16}$$

As a matrix, this is the conjugate transpose of the matrix for A:

$$A^\dagger = (A^*)^T = \begin{pmatrix} 0 & 0 & 0 \\ 1-i & -i & 0 \\ 1+i & 1 & 0 \end{pmatrix}. \tag{17}$$

Interpretation: if $A = |\psi\rangle\langle\phi|$ maps $|v\rangle \mapsto |\psi\rangle\langle\phi|v\rangle$ (extract overlap with $|\phi\rangle$ and output along $|\psi\rangle$), then $A^\dagger = |\phi\rangle\langle\psi|$ maps $|v\rangle \mapsto |\phi\rangle\langle\psi|v\rangle$ (extract overlap with $|\psi\rangle$ and output along $|\phi\rangle$).

Problem 2: Orthogonality, normalization, bases, and transformations (J. Gulla, O. Leinonen)

Throughout this exercise we work in the computational basis $\{|0\rangle, |1\rangle\}$. Consider the two (unnormalized) vectors

$$|v\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad |w\rangle = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{18}$$

a) Show that $|v\rangle$ and $|w\rangle$ are orthogonal. Find the normalized vectors $|v_\mathrm{N}\rangle$ and $|w_\mathrm{N}\rangle$.

Solution: Two vectors are orthogonal if their inner product is zero:

$$\langle w|v\rangle = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 - 1 = 0. \tag{19}$$

Their norms are

$$\| |v\rangle \| = \sqrt{\langle v|v\rangle} = \sqrt{\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \sqrt{2},$$
$$\| |w\rangle \| = \sqrt{\langle w|w\rangle} = \sqrt{\begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \sqrt{2}.$$

Thus the normalized vectors are

$$|v_\mathrm{N}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad |w_\mathrm{N}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{20}$$

We usually denote $|v_\mathrm{N}\rangle = |+\rangle$ and $|w_\mathrm{N}\rangle = |-\rangle$.

b) Show that $\{|v_\mathrm{N}\rangle, |w_\mathrm{N}\rangle\}$ is an orthonormal basis. Verify the completeness relation (resolution of the identity)

$$|v_\mathrm{N}\rangle\langle v_\mathrm{N}| + |w_\mathrm{N}\rangle\langle w_\mathrm{N}| = \mathbb{1}. \tag{21}$$

Solution: We already have $\langle v_\mathrm{N}|v_\mathrm{N}\rangle = \langle w_\mathrm{N}|w_\mathrm{N}\rangle = 1$ and $\langle w_\mathrm{N}|v_\mathrm{N}\rangle = 0$, so $\{|v_\mathrm{N}\rangle, |w_\mathrm{N}\rangle\}$ is orthonormal. In a two-dimensional space, any orthonormal set of two vectors is automatically a basis.

To verify completeness, compute the two rank-one projectors:

$$|v_\mathrm{N}\rangle\langle v_\mathrm{N}| = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$
$$|w_\mathrm{N}\rangle\langle w_\mathrm{N}| = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Adding them gives

$$|v_\mathrm{N}\rangle\langle v_\mathrm{N}| + |w_\mathrm{N}\rangle\langle w_\mathrm{N}| = \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \mathbb{1}, \tag{22}$$

which is exactly (21).

c) Use (21) to express $|0\rangle$ and $|1\rangle$ as linear combinations of $|v_{\mathrm{N}}\rangle$ and $|w_{\mathrm{N}}\rangle$. Next, define bases $\mathcal{B} = \{|0\rangle, |1\rangle\}$ and $\mathcal{B}' = \{|v_{\mathrm{N}}\rangle, |w_{\mathrm{N}}\rangle\}$. For any state $|\psi\rangle$, define coordinate column vectors $[\psi]_{\mathcal{B}}$ and $[\psi]_{\mathcal{B}'}$ by

$$|\psi\rangle = \big([\psi]_{\mathcal{B}}\big)_0 |0\rangle + \big([\psi]_{\mathcal{B}}\big)_1 |1\rangle = \big([\psi]_{\mathcal{B}'}\big)_0 |v_{\mathrm{N}}\rangle + \big([\psi]_{\mathcal{B}'}\big)_1 |w_{\mathrm{N}}\rangle. \tag{23}$$

The *change-of-basis matrix* U from $\mathcal{B}'$ to $\mathcal{B}$ is defined by

$$[\psi]_{\mathcal{B}} = U\,[\psi]_{\mathcal{B}'} \qquad \text{for all } |\psi\rangle. \tag{24}$$

Find U. Hint[1].

Solution: Insert the identity in the form (21):

$$|0\rangle = \mathbb{1}\,|0\rangle = |v_{\mathrm{N}}\rangle\langle v_{\mathrm{N}}|0\rangle + |w_{\mathrm{N}}\rangle\langle w_{\mathrm{N}}|0\rangle, \qquad |1\rangle = \mathbb{1}\,|1\rangle = |v_{\mathrm{N}}\rangle\langle v_{\mathrm{N}}|1\rangle + |w_{\mathrm{N}}\rangle\langle w_{\mathrm{N}}|1\rangle. \tag{25}$$

Compute the coefficients:

$$\langle v_{\mathrm{N}}|0\rangle = \frac{1}{\sqrt{2}}, \quad \langle w_{\mathrm{N}}|0\rangle = \frac{1}{\sqrt{2}}, \qquad \langle v_{\mathrm{N}}|1\rangle = \frac{1}{\sqrt{2}}, \quad \langle w_{\mathrm{N}}|1\rangle = -\frac{1}{\sqrt{2}}. \tag{26}$$

Hence

$$|0\rangle = \frac{1}{\sqrt{2}}\big(|v_{\mathrm{N}}\rangle + |w_{\mathrm{N}}\rangle\big), \qquad |1\rangle = \frac{1}{\sqrt{2}}\big(|v_{\mathrm{N}}\rangle - |w_{\mathrm{N}}\rangle\big). \tag{27}$$

To find U, apply $[\psi]_{\mathcal{B}} = U[\psi]_{\mathcal{B}'}$ to the basis vectors of $\mathcal{B}'$. Since $[v_{\mathrm{N}}]_{\mathcal{B}'} = \begin{pmatrix}1\\0\end{pmatrix}$ and $[w_{\mathrm{N}}]_{\mathcal{B}'} = \begin{pmatrix}0\\1\end{pmatrix}$, we get

$$U\begin{pmatrix}1\\0\end{pmatrix} = [v_{\mathrm{N}}]_{\mathcal{B}}, \qquad U\begin{pmatrix}0\\1\end{pmatrix} = [w_{\mathrm{N}}]_{\mathcal{B}}. \tag{28}$$

Therefore the columns of U are the coordinate vectors of $|v_{\mathrm{N}}\rangle$ and $|w_{\mathrm{N}}\rangle$ in the basis $\mathcal{B}$:

$$U = \begin{pmatrix} | & | \\ [v_{\mathrm{N}}]_{\mathcal{B}} & [w_{\mathrm{N}}]_{\mathcal{B}} \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ |v_{\mathrm{N}}\rangle & |w_{\mathrm{N}}\rangle \\ | & | \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \tag{29}$$

Thus U is the Hadamard matrix H.

d) Show that the change-of-basis matrix U you found is unitary. Interpret $U^\dagger$ in terms of changing coordinates between $\mathcal{B}$ and $\mathcal{B}'$.

Solution: Since $\mathcal{B}'$ is an orthonormal basis, the columns of U are orthonormal, so $U^\dagger U = \mathbb{1}$. Concretely, with $U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ we compute

$$U^\dagger U = U^T U = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \mathbb{1}. \tag{30}$$

Thus U is unitary and $U^{-1} = U^\dagger$.

In coordinates, U converts $\mathcal{B}'$-coordinates to $\mathcal{B}$-coordinates:

$$[\psi]_{\mathcal{B}} = U[\psi]_{\mathcal{B}'}. \tag{31}$$

[1] For the change-of-basis matrix U, compute where it sends the coordinate vectors $[v_{\mathrm{N}}]_{\mathcal{B}'}$ and $[w_{\mathrm{N}}]_{\mathcal{B}'}$.

Multiplying by $U^\dagger$ gives the reverse transformation:

$$[\psi]_{\mathcal{B}'} = U^\dagger [\psi]_{\mathcal{B}}. \tag{32}$$

e) The relation $[\psi]_{\mathcal{B}} = U[\psi]_{\mathcal{B}'}$ describes a *passive* change of coordinates (same state, new basis). In contrast, an *active* unitary maps the state to a new state $|\psi\rangle \mapsto U|\psi\rangle$, whose coordinates in the *same* basis satisfy $[U|\psi\rangle]_{\mathcal{B}} = U[\psi]_{\mathcal{B}}$. Explain in 1–2 sentences what is held fixed in each case.

Solution: Passive: the abstract state $|\psi\rangle$ is held fixed, while we change the basis used to describe it (so the coordinate column changes from $[\psi]_{\mathcal{B}'}$ to $[\psi]_{\mathcal{B}}$).

Active: the basis is held fixed (here $\mathcal{B}$), while the state changes from $|\psi\rangle$ to $U|\psi\rangle$ (so the coordinate column changes from $[\psi]_{\mathcal{B}}$ to $[U|\psi\rangle]_{\mathcal{B}}$).

Numerically, the same matrix U can appear in both formulas, but it plays different roles: in the *passive* relation it converts coordinate columns between the two bases for the *same* abstract state, while in the *active* relation it represents a unitary acting on the *state* in a fixed basis.

Problem 3: Projectors and changing basis for operators (J. Gulla)

Unless otherwise stated, we work in the computational basis $\{|0\rangle, |1\rangle\}$. Let

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \qquad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{33}$$

Note: $\{|+\rangle, |-\rangle\}$ is the same basis as $\{|v_{\mathrm{N}}\rangle, |w_{\mathrm{N}}\rangle\}$ from Problem 2, so the same Hadamard matrix H will reappear below.

a) Compute the projector $P = |+\rangle\langle+|$ as a 2×2 matrix in the computational basis. Verify that $P^2 = P$ and that $P^\dagger = P$.

Solution: We have

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \qquad \langle+| = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\end{pmatrix}. \tag{34}$$

Thus

$$[P]_{\{|0\rangle,|1\rangle\}} = |+\rangle\langle+| = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}. \tag{35}$$

Then

$$P^2 = \left(\frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\right)^2 = \frac{1}{4}\begin{pmatrix}2 & 2\\2 & 2\end{pmatrix} = P, \tag{36}$$

so P is a projector. Since the matrix is real symmetric, $P^\dagger = P$.

b) Compute $P|+\rangle$ and $P|-\rangle$ and interpret the result geometrically. Conclude what the eigenvalues of P are.

Solution: Using $P = |+\rangle\langle+|$,

$$P|+\rangle = |+\rangle\langle+|+\rangle = |+\rangle, \qquad P|-\rangle = |+\rangle\langle+|-\rangle = \mathbf{0}. \tag{37}$$

Here $\mathbf{0}$ denotes the zero vector (not the state $|0\rangle$).

Thus P projects onto $\mathrm{span}\{|+\rangle\}$ and annihilates the orthogonal direction $|-\rangle$. In particular, $|+\rangle$ is an eigenvector with eigenvalue 1 and $|-\rangle$ is an eigenvector with eigenvalue 0. Since we have

found two linearly independent eigenvectors, and P acts on a two-dimensional vector space, there cannot be any additional eigenvalues. So the eigenvalues of P are $\{1, 0\}$.

c) Let $\mathcal{B}_\pm = \{|+\rangle, |-\rangle\}$. Compute the matrix representation $[P]_{\mathcal{B}_\pm}$ of P in the basis $\mathcal{B}_\pm$. Compare it with the matrix representation of P in the computational basis. Hint[2].

Solution: Using $P|+\rangle = |+\rangle$ and $P|-\rangle = \mathbf{0}$,

$$[P]_{\mathcal{B}_\pm} = \begin{pmatrix} \langle +|P|+\rangle & \langle +|P|-\rangle \\ \langle -|P|+\rangle & \langle -|P|-\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{38}$$

In contrast, in the computational basis we found

$$[P]_{\{|0\rangle,|1\rangle\}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{39}$$

Thus the *same* operator has different matrix representations in different bases.

d) Let U be the change-of-basis matrix from $\mathcal{B}_\pm$ to the computational basis. (This is exactly the matrix $U = H$ found in Problem 2.) Since we found that U is unitary, verify that

$$[P]_{\mathcal{B}_\pm} = U^\dagger [P]_{\{|0\rangle,|1\rangle\}} U. \tag{40}$$

(*General rule:* for any operator A, one has $[A]_{\mathcal{B}'} = S^{-1}[A]_{\mathcal{B}}S$ for a change-of-basis matrix S.)

Solution: Using $U = H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $[P]_{\{|0\rangle,|1\rangle\}} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ from (a), we compute

$$U^\dagger [P]_{\{|0\rangle,|1\rangle\}} U = H^\dagger \left(\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right) H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = [P]_{\mathcal{B}_\pm},$$

which verifies (40).

e) Let $Q = \mathbb{1} - P$. Show that $Q = |-\rangle\langle -|$ and that $PQ = 0$, and verify the completeness relation

$$P + Q = \mathbb{1}. \tag{41}$$

Solution: Compute

$$Q = \mathbb{1} - P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{42}$$

But

$$|-\rangle\langle -| = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{43}$$

so indeed $Q = |-\rangle\langle -|$.

Moreover,

$$PQ = |+\rangle\langle +|\,|-\rangle\langle -| = |+\rangle\langle +|-\rangle\langle -| = 0. \tag{44}$$

[2] Compute matrix elements $([P]_{\mathcal{B}_\pm})_{ij} = \langle b_i|P|b_j\rangle$ where $\mathcal{B}_\pm = \{|b_0\rangle, |b_1\rangle\} = \{|+\rangle, |-\rangle\}$.

Finally, we see directly from the definition of Q that $P + Q = |+\rangle\langle+| + |-\rangle\langle-| = \mathbb{1}$ is the resolution of the identity in the $\pm$ basis.

Problem 4: Pauli matrices **(Adapted from Mike & Ike 2.11)**

The Pauli matrices are defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{45}$$

Throughout this exercise we work in the computational basis $\{|0\rangle, |1\rangle\}$.

a) Write each Pauli matrix as a sum of outer products in the computational basis, i.e.

$$\sigma = \sum_{i,j\in\{0,1\}} \left(\langle i|\sigma\,|j\rangle\right)\, |i\rangle\langle j|. \tag{46}$$

Solution: Using

$$|0\rangle\langle 0| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad |0\rangle\langle 1| = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad |1\rangle\langle 0| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad |1\rangle\langle 1| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{47}$$

we obtain

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \qquad \sigma_y = -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|, \qquad \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|. \tag{48}$$

b) Find the eigenvalues and a set of normalized eigenvectors for each of $\sigma_x, \sigma_y, \sigma_z$. Hint[3].

Solution: For Pauli matrices σ there is a shortcut: since $\sigma^2 = \mathbb{1}$, any eigenvalue λ must satisfy $\lambda^2 = 1$, hence $\lambda = \pm 1$. A convenient choice of normalized eigenvectors is:

$$\begin{aligned}
\sigma_x\,|0_x\rangle &= +\,|0_x\rangle, & |0_x\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle, \\
\sigma_x\,|1_x\rangle &= -\,|1_x\rangle, & |1_x\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle, \\
\sigma_y\,|0_y\rangle &= +\,|0_y\rangle, & |0_y\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle), \\
\sigma_y\,|1_y\rangle &= -\,|1_y\rangle, & |1_y\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle), \\
\sigma_z\,|0_z\rangle &= +\,|0_z\rangle, & |0_z\rangle &= |0\rangle, \\
\sigma_z\,|1_z\rangle &= -\,|1_z\rangle, & |1_z\rangle &= |1\rangle.
\end{aligned}$$

(Any overall phase factor on an eigenvector is also allowed.)

[3] For some matrix σ, use the standard eigenvalue method: solve the characteristic equation $\det(\sigma - \lambda\mathbb{1}) = 0$ to find the eigenvalues λ. For each eigenvalue, solve $(\sigma - \lambda\mathbb{1})\,|v\rangle = \mathbf{0}$ to find an eigenvector $|v\rangle$, and then normalize it.

c) Use your eigenpairs to write the spectral decomposition

$$\sigma = \sum_k \lambda_k \, |v_k\rangle\langle v_k|, \tag{49}$$

for each Pauli matrix σ. Explain briefly the difference between (46) and (49).

Solution:

$$\sigma_x = (+1)\,|0_x\rangle\langle 0_x| + (-1)\,|1_x\rangle\langle 1_x| = |0_x\rangle\langle 0_x| - |1_x\rangle\langle 1_x| = |+\rangle\langle +| - |-\rangle\langle -|,$$
$$\sigma_y = |0_y\rangle\langle 0_y| - |1_y\rangle\langle 1_y|,$$
$$\sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1|.$$

Here (46) is an expansion in a fixed basis (the computational basis), while (49) is an expansion in the (different) eigenbases of each Pauli matrix σ.

d) State what the matrix representation of each $\sigma_x, \sigma_y, \sigma_z$ looks like in its own eigenbasis.

Solution: In the ordered eigenbasis $\{|0_\alpha\rangle, |1_\alpha\rangle\}$ of σ_α (for $\alpha = x, y, z$), the operator is diagonal with entries $+1$ and -1:

$$[\sigma_\alpha]_{\{|0_\alpha\rangle, |1_\alpha\rangle\}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{50}$$

e) Use the spectral decomposition to express the projectors onto the ± 1 eigenspaces of each Pauli matrix σ as simple linear combinations of $\mathbb{1}$ and σ. Verify that these two projectors sum to the identity.

Solution: From the spectral decomposition for a fixed α,

$$\sigma_\alpha = |0_\alpha\rangle\langle 0_\alpha| - |1_\alpha\rangle\langle 1_\alpha|, \qquad \mathbb{1} = |0_\alpha\rangle\langle 0_\alpha| + |1_\alpha\rangle\langle 1_\alpha|. \tag{51}$$

Solving these gives

$$|0_\alpha\rangle\langle 0_\alpha| = \frac{1}{2}(\mathbb{1} + \sigma_\alpha), \qquad |1_\alpha\rangle\langle 1_\alpha| = \frac{1}{2}(\mathbb{1} - \sigma_\alpha), \tag{52}$$

and indeed their sum is $\mathbb{1}$.

Problem 5: Unitary matrices (J. Gulla)

a) Let U be unitary. Show that unitaries preserve inner products:

$$\langle U\psi | U\phi\rangle = \langle \psi | \phi\rangle \qquad \text{for all } |\psi\rangle, |\phi\rangle. \tag{53}$$

Conclude that unitaries preserve norms.

Solution: We have

$$\langle U\psi | U\phi\rangle = \langle \psi | U^\dagger U \, |\phi\rangle = \langle \psi | \mathbb{1} \, |\phi\rangle = \langle \psi | \phi\rangle. \tag{54}$$

Setting $|\phi\rangle = |\psi\rangle$ gives $\|U\,|\psi\rangle\|^2 = \langle\psi|U^\dagger U\,|\psi\rangle = \langle\psi|\psi\rangle = \|\,|\psi\rangle\|^2$, so norms are preserved.

b) Show that all eigenvalues of a unitary matrix have modulus 1, i.e. can be written as $e^{i\theta}$ for some real θ. Hint[4].

Solution: Let $|v\rangle \neq 0$ be an eigenvector of a unitary matrix U with eigenvalue λ:

$$U|v\rangle = \lambda|v\rangle. \tag{55}$$

Taking norms, unitarity gives $\|U|v\rangle\| = \||v\rangle\|$, while $\|\lambda|v\rangle\| = |\lambda|\,\||v\rangle\|$. Thus $\||v\rangle\| = |\lambda|\,\||v\rangle\|$. Since $\||v\rangle\| > 0$, we conclude $|\lambda| = 1$. Therefore λ lies on the unit circle and can be written $\lambda = e^{i\theta}$ for some real θ.

Problem 6: Commutators (J. Bergli)

We use the standard notation for commutators and anticommutators

$$[A, B] = AB - BA, \qquad \{A, B\} = AB + BA, \tag{56}$$

where A and B are operators or matrices. Show the following identities.

a) Show that

$$[A, BC] = [A, B]\,C + B\,[A, C]. \tag{57}$$

Solution:

$$[A, BC] = ABC - BCA = ABC - BAC + BAC - BCA = [A, B]C + B[A, C]. \tag{58}$$

b) Prove that $(AB)^\dagger = B^\dagger A^\dagger$. Use this to show that

$$[A, B]^\dagger = [B^\dagger, A^\dagger], \qquad \{A, B\}^\dagger = \{B^\dagger, A^\dagger\}. \tag{59}$$

Solution: For matrices, $(AB)^\dagger$ is the conjugate transpose:

$$(AB)^\dagger = (AB)^{*T} = B^{*T}A^{*T} = B^\dagger A^\dagger. \tag{60}$$

Then

$$[A, B]^\dagger = (AB - BA)^\dagger = (AB)^\dagger - (BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = [B^\dagger, A^\dagger], \tag{61}$$

and similarly

$$\{A, B\}^\dagger = (AB + BA)^\dagger = B^\dagger A^\dagger + A^\dagger B^\dagger = \{B^\dagger, A^\dagger\}. \tag{62}$$

[4] If $U|v\rangle = \lambda|v\rangle$, compare $\|U|v\rangle\|$ and $\|\lambda|v\rangle\|$.