University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 1

This problem set will be discussed in the group exercise session on **Jan. 28**.

Problem 1: Trace and Pauli identities (J. Gulla, O. Leinonen)

Throughout this problem we work in the computational basis $\{|0\rangle, |1\rangle\}$ and use

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

You may use that the trace is cyclic, $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, as we proved in the lectures.

a) Trace of Pauli matrices. Compute $\mathrm{Tr}(\mathbb{1})$ and show that $\mathrm{Tr}(\sigma_x) = \mathrm{Tr}(\sigma_y) = \mathrm{Tr}(\sigma_z) = 0$.

b) Linearity. Let A, B be arbitrary matrices and $a, b \in \mathbb{C}$. Show that

$$\mathrm{Tr}(aA + bB) = a\ \mathrm{Tr}(A) + b\ \mathrm{Tr}(B). \tag{2}$$

c) Outer products in the trace. Let $|\phi\rangle, |\psi\rangle$ be vectors.

(i) Show that $\mathrm{Tr}(|\phi\rangle\langle\psi|) = \langle\psi|\phi\rangle$.

(ii) Show that for any matrix A, we have $\mathrm{Tr}(A\,|\phi\rangle\langle\psi|) = \langle\psi|A\,|\phi\rangle$.

d) Pauli algebra. Recall from Problem set 0 that the Pauli matrices can be written in the form:

$$\begin{aligned} \mathbb{1} &= |0\rangle\langle 0| + |1\rangle\langle 1|, \\ \sigma_x &= |0\rangle\langle 1| + |1\rangle\langle 0|, \\ \sigma_y &= -i\,|0\rangle\langle 1| + i\,|1\rangle\langle 0|, \\ \sigma_z &= |0\rangle\langle 0| - |1\rangle\langle 1|. \end{aligned} \tag{3}$$

(i) Use these expressions to show $\sigma_i^\dagger = \sigma_i$ and $\sigma_i^2 = \mathbb{1}$ for $i \in \{x, y, z\}$.

(ii) Compute the three cyclic permutations:

$$\sigma_x\sigma_y = i\sigma_z, \qquad \sigma_y\sigma_z = i\sigma_x, \qquad \sigma_z\sigma_x = i\sigma_y. \tag{4}$$

Conclude the compact product rule

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1} + i \sum_{k\in\{x,y,z\}} \varepsilon_{ijk}\sigma_k, \qquad \text{for } i, j \in \{x, y, z\}, \tag{5}$$

where ε_{ijk} is totally antisymmetric (changes sign when any two indices are swapped and is 0 if any two indices coincide) and $\varepsilon_{xyz} = +1$. Deduce in particular that $\sigma_i\sigma_j = -\sigma_j\sigma_i$ for $i \neq j$, and that

$$[\sigma_i, \sigma_j] = 2i \sum_k \varepsilon_{ijk}\sigma_k, \qquad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}. \tag{6}$$

Problem 2: Bloch sphere rotations (J. Gulla, O. Leinonen)

Throughout this problem we work in the computational basis $\{|0\rangle, |1\rangle\}$ and use $\sigma_x, \sigma_y, \sigma_z$. Consider the normalized qubit state

$$|\psi\rangle = \frac{\sqrt{3}}{2}|0\rangle + \frac{e^{i\pi/3}}{2}|1\rangle = \begin{pmatrix} \sqrt{3}/2 \\ e^{i\pi/3}/2 \end{pmatrix}. \tag{7}$$

The Bloch vector of a pure state is defined by

$$\vec{r}(\psi) = \langle\psi|\vec{\sigma}|\psi\rangle = \big(\langle\psi|\sigma_x|\psi\rangle, \langle\psi|\sigma_y|\psi\rangle, \langle\psi|\sigma_z|\psi\rangle\big), \qquad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z). \tag{8}$$

a) Bloch angles and vector.

(i) Explain briefly why any normalized qubit state can be written as

$$|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \qquad \text{for } \theta \in [0,\pi],\ \phi \in [0, 2\pi), \tag{9}$$

up to a global phase. Find (θ, ϕ) for (7).

(ii) Compute $\vec{r}(\theta,\phi)$ and show

$$\vec{r}(\theta,\phi) = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta). \tag{10}$$

Use this to compute $\vec{r}(\psi)$ for (7), and verify that $\|\vec{r}(\psi)\| = 1$. Hint[1].

b) Rotation about x and z axes. For parameters $\alpha, \beta \in \mathbb{R}$, define

$$U_x(\alpha) = \exp\left(-i\frac{\alpha}{2}\sigma_x\right), \qquad U_z(\beta) = \exp\left(-i\frac{\beta}{2}\sigma_z\right). \tag{11}$$

(i) You may use that σ_x has eigenvectors

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle), \qquad \sigma_x|\pm\rangle = \pm|\pm\rangle. \tag{12}$$

Use the spectral theorem to show

$$U_x(\alpha) = e^{-i\alpha/2}|+\rangle\langle+| + e^{i\alpha/2}|-\rangle\langle-|, \tag{13}$$

and simplify this to the closed form

$$U_x(\alpha) = \cos\frac{\alpha}{2}\,\mathbb{1} - i\sin\frac{\alpha}{2}\,\sigma_x. \tag{14}$$

Write $U_x(\alpha)$ as a 2×2 matrix in the computational basis.

(ii) Show directly, using that σ_z is diagonal, that

$$U_z(\beta) = \begin{pmatrix} e^{-i\beta/2} & 0 \\ 0 & e^{i\beta/2} \end{pmatrix} = \cos\frac{\beta}{2}\,\mathbb{1} - i\sin\frac{\beta}{2}\,\sigma_z. \tag{15}$$

[1] Use $e^{i\phi} + e^{-i\phi} = 2\cos\phi$ and $e^{i\phi} - e^{-i\phi} = 2i\sin\phi$, and the double-angle identities $2\cos(\theta/2)\sin(\theta/2) = \sin\theta$ and $\cos^2(\theta/2) - \sin^2(\theta/2) = \cos\theta$.

c) Rotation about an arbitrary axis. Let $\hat{n} = (n_x, n_y, n_z) \in \mathbb{R}^3$ be a unit vector and define

$$\sigma_{\hat{n}} = \hat{n} \cdot \vec{\sigma} = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z. \tag{16}$$

Show that $\sigma_{\hat{n}}$ is Hermitian and that $\sigma_{\hat{n}}^2 = \mathbb{1}$. Using the power-series definition of the exponential, show that

$$U(\hat{n}, \alpha) = \exp\Big(-i\frac{\alpha}{2}\sigma_{\hat{n}}\Big) = \cos\frac{\alpha}{2}\,\mathbb{1} - i\sin\frac{\alpha}{2}\,\sigma_{\hat{n}}. \tag{17}$$

Show that $U(\hat{n}, \alpha)$ is unitary. Hint[2].

d) z rotations shift ϕ. Let

$$U_z(\beta) = e^{-i\beta\sigma_z/2} \qquad \text{and} \qquad |\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle. \tag{18}$$

(i) Show that

$$U_z(\beta)\,|\psi(\theta,\phi)\rangle \sim |\psi(\theta,\phi+\beta)\rangle, \tag{19}$$

where $\sim$ means equality up to a global phase.

(ii) Use this and $\vec{r}(\theta,\phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ to show that $U_z(\beta)$ rotates the Bloch vector about the z axis by angle β:

$$\begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix} \mapsto R_z(\beta)\begin{pmatrix} r_x \\ r_y \\ r_z \end{pmatrix}, \qquad R_z(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

e) Larmor precession. Consider a spin-$\frac{1}{2}$ particle in a uniform magnetic field $\vec{B} = B\,\hat{z}$ with Hamiltonian

$$H = -\frac{\hbar\gamma}{2}\,\vec{B}\cdot\vec{\sigma} = -\frac{\hbar\omega}{2}\sigma_z, \qquad \omega = \gamma B. \tag{21}$$

Let $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$ with $U(t) = e^{-iHt/\hbar}$ and define $\vec{r}(t)$ from $|\psi(t)\rangle$.

(i) Show that

$$U(t) = e^{-iHt/\hbar} = e^{\,i\omega t\,\sigma_z/2} = U_z(-\omega t). \tag{22}$$

(ii) Use the previous subproblem to show that if $|\psi(0)\rangle = |\psi(\theta_0,\phi_0)\rangle$, then

$$|\psi(t)\rangle \sim |\psi(\theta_0, \phi_0 - \omega t)\rangle. \tag{23}$$

Write down $(r_x(t), r_y(t), r_z(t))$ in terms of $(r_x(0), r_y(0), r_z(0))$.

(iii) Show that $\vec{r}(t)$ satisfies the Bloch equation

$$\dot{\vec{r}}(t) = -\omega\,\hat{z} \times \vec{r}(t). \tag{24}$$

[2] Use the properties of the Pauli matrices found in Problem 1. Expand $(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2$ and see that cross terms cancel in pairs because $\sigma_i\sigma_j + \sigma_j\sigma_i = 0$ for $i \neq j$. For the power series, split it into even and odd powers.

Problem 3: Classical computing, universality, reversibility, and dual-rail encoding (J. Gulla)

In this problem we review a few key ideas from classical circuit complexity and reversible computation, and connect them to the constraints of quantum gates (unitaries). In the last parts we also consider a simple redundancy encoding, dual-rail, to practice the viewpoint that logical information can live in a constrained subset, a code space, of a larger physical system.

We use standard Boolean notation throughout:

- Bits are elements of $\{0, 1\}$, where 0 means false and 1 means true.
- Negation is written $\overline{x}$ and corresponds to the gate $\mathrm{NOT}(x)$.
- Conjunction is written $x \wedge y$ and corresponds to $\mathrm{AND}(x, y)$.
- Disjunction is written $x \vee y$ and corresponds to $\mathrm{OR}(x, y)$.
- Exclusive OR (XOR) is written $x \oplus y$ and means addition modulo 2.

A gate set is called universal if it can be used to implement any Boolean function as a finite circuit (with wires, and with COPY or constants when allowed).

a) NAND with COPY. The NAND gate is

$$\mathrm{NAND}(x, y) = \overline{x \wedge y}. \tag{25}$$

Assume you have access to fan-out, i.e. a COPY operation, that can be applied to any bit on a wire. Show that the gate set $\{\mathrm{NAND}, \mathrm{COPY}\}$ can generate NOT, AND, and OR without using external constant inputs. Why does this imply that $\{\mathrm{NAND}, \mathrm{COPY}\}$ is universal for classical computation even though it only contains a single nontrivial gate, NAND? Hint[3].

b) NAND without COPY. Now assume you do not have access to a COPY gate as a primitive operation. Do *not* use the shortcut $\mathrm{NOT}(x) = \mathrm{NAND}(x, x)$. Instead, you are allowed access to constant inputs 0 and 1 as many times as you need. Show that NAND together with constants can generate NOT, AND, and OR without using COPY. Which step is the one that uses the constant 1, and why does this again imply universality?

c) Reversibility.

(i) Explain briefly why quantum gates are always reversible.

(ii) Show that the classical NAND gate is irreversible, and explain why this prevents a direct implementation of NAND as a quantum gate.

d) Toffoli simulates NAND. The Toffoli gate acts on three bits:

$$\mathrm{Toffoli}(a, b, c) = (a, b, c \oplus (a \wedge b)). \tag{26}$$

(i) Show that Toffoli is reversible by verifying $\mathrm{Toffoli}^2 = \mathbb{1}$.

(ii) Show that NAND can be simulated using Toffoli by fixing $c = 1$. What is the third output bit?

(iii) What does this imply about the relationship between classical and quantum computation?

[3] Use De Morgan's laws: $\overline{x \wedge y} = \overline{x} \vee \overline{y}$ and $\overline{x \vee y} = \overline{x} \wedge \overline{y}$.

e) Dual-rail encoding. Consider the dual-rail encoding of one logical bit into two physical bits, where valid strings have exactly one 1:

$$\widehat{0} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \widehat{1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{27}$$

(i) List the two invalid physical strings.

(ii) Show that a physical SWAP of the two wires implements a logical $\widehat{\text{NOT}}$ on valid codewords.

(iii) Show that any single physical bit-flip error maps a valid codeword to an invalid string. Explain briefly why this means the encoding can detect (although not correct) a single bit flip.

f) Logic gates on dual-rail. Assume you can apply the physical gates $\{\text{AND}, \text{OR}, \text{COPY}\}$ to the physical bits, and you can also apply physical SWAP, which acts as logical $\widehat{\text{NOT}}$ on codewords. Show that you can implement a logical $\widehat{\text{AND}}$ on encoded bits:

$$(\widehat{x}, \widehat{y}) \mapsto \widehat{x \wedge y}. \tag{28}$$

Conclude that $\widehat{\text{NOT}}$ and $\widehat{\text{AND}}$ form a universal logical gate set on the encoded bits. What does universality mean here in practice: how would you build a circuit for an arbitrary Boolean function of the *logical* bits using only operations on the *physical* bits? Hint[4].

[4] Write $\widehat{x} = (x_0, x_1)$, $\widehat{y} = (y_0, y_1)$. Try $z_0 = x_0 \wedge y_0$ and use De Morgan to express $\overline{z_0}$ in terms of x_1, y_1. Find physical gates that output $z_0 = x_0 \wedge y_0$ and $z_1 = \overline{z_0}$.