University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 2

This problem set will be discussed in the group exercise session on **Feb. 4**.

Problem 1: Two-qubit tensor products (O. Leinonen)

In this exercise you will practice tensor products of operators and states, and inner products on two qubits. We use the computational basis

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}. \tag{1}$$

a) Tensor products. Compute the matrix representations in the computational basis of

$$\mathbb{1} \otimes X, \qquad X \otimes \mathbb{1}, \qquad X \otimes Z, \tag{2}$$

where

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Is the tensor product commutative? Hint[1].

b) Product states as 4-vectors. Let $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Compute each of the following as a linear combination of $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ and as a 4-component column vector in the computational basis:

$$|+0\rangle = |+\rangle \otimes |0\rangle, \qquad |0+\rangle = |0\rangle \otimes |+\rangle, \qquad |++\rangle = |+\rangle \otimes |+\rangle. \tag{4}$$

c) Inner products. Using either Dirac notation or your 4-vector representations above, compute

$$\langle +0|0+\rangle, \qquad \langle +0|++\rangle, \qquad \langle 0+|++\rangle. \tag{5}$$

Verify that $|+0\rangle, |0+\rangle, |++\rangle$ are normalized, and determine which pairs (if any) are orthogonal.

Problem 2: Adjoint and trace of tensor products for general matrices (J. Gulla, O. Leinonen)

a) Matrix elements of the adjoint. You have seen that the adjoint of a linear operator M satisfies

$$\langle \phi | M^\dagger \psi \rangle = \langle M\phi | \psi \rangle \tag{6}$$

for all vectors $|\phi\rangle, |\psi\rangle$. Use (6) to show that for any orthonormal basis $\{|r\rangle\}$,

$$\langle r | M^\dagger | s \rangle = \langle s | M | r \rangle^*. \tag{7}$$

[1] For 2x2 matrices A and B, if $A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$, then $A \otimes B = \begin{pmatrix} a_{00}B & a_{01}B \\ a_{10}B & a_{11}B \end{pmatrix}$.

b) Adjoint distributes. Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ be square matrices. Let $\{|i\rangle\}$ be an orthonormal basis of $\mathbb{C}^m$ and $\{|j\rangle\}$ an orthonormal basis of $\mathbb{C}^n$. Define the product basis

$$|i,j\rangle = |i\rangle \otimes |j\rangle, \tag{8}$$

with the resolution of the identity

$$\mathbb{1}_m \otimes \mathbb{1}_n = \sum_i \sum_j |i,j\rangle\langle i,j|. \tag{9}$$

i) Show that

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger. \tag{10}$$

Hint[2].

ii) Deduce that:

- if A and B are unitary, then $A \otimes B$ is unitary;
- if A and B are Hermitian, then $A \otimes B$ is Hermitian.

c) Trace factorizes. Show that for any $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$,

$$\mathrm{Tr}(A \otimes B) = \mathrm{Tr}(A)\ \mathrm{Tr}(B). \tag{11}$$

Problem 3: Quantum gate identities (O. Leinonen)

a) The Hadamard gate is

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{12}$$

Show that $H = \frac{1}{\sqrt{2}}(X + Z)$ and use this to prove that $HXH = Z$ and $HZH = X$. Hint[3].

b) The S and T gates are:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}. \tag{13}$$

i) Show that $T^2 = S$, $S^2 = Z$, and therefore $T^8 = \mathbb{1}$.

ii) Show that

$$S = e^{i\pi/4}\, e^{-i\pi Z/4}, \tag{14}$$

meaning that S is a rotation about the Z-axis (up to global phase).

iii) Let $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = iXZ$ and show that $SXS^\dagger = Y$ and $SZS^\dagger = Z$. Hint[4].

Problem 4: Control and target qubits in CNOT (J. Gulla, O. Leinonen)

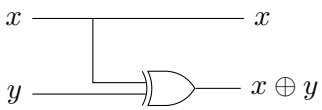
The goal of this problem is to understand and be able to predict what happens in circuits containing CNOT gates. Throughout, CNOT_{12} means a CNOT with control on qubit 1 and target on qubit 2.

[2] Compare matrix elements in the product basis and use (7) applied to $\{|i,j\rangle\}$. To evaluate $\langle k,\ell|(A\otimes B)\,|i,j\rangle$ you may insert resolutions of the identity on each subsystem, e.g. $\mathbb{1}_m = \sum_i |i\rangle\langle i|$ and $\mathbb{1}_n = \sum_j |j\rangle\langle j|$.

[3] Use $X^2 = Z^2 = \mathbb{1}$ and $XZ = -ZX$.

[4] For the rotation, recall that $e^{-i\frac{\theta}{2}Z} = \cos\frac{\theta}{2}\mathbb{1} - i\sin\frac{\theta}{2}Z$.

a) Implementing a classical controlled-XOR. Consider the following classical circuit on bits x and y:



This implements an XOR on y controlled by x, while keeping x unchanged:

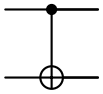
$$(x, y) \mapsto (x, x \oplus y). \tag{15}$$

Let us see what happens when we implement such a circuit on a quantum computer.

i) Treating the computational basis labels as classical bits $x, y \in \{0, 1\}$, translate the action of (15) as a basis state update map. Hint[5].

ii) Use your answer to write down a 4×4 matrix representing this quantum gate in the computational basis. We call this the CNOT_{12} gate, where the control (qubit 1) is unchanged and the target (qubit 2) flips if and only if qubit 1 is $|1\rangle$. It has the quantum circuit representation:

$$\mathrm{CNOT}_{12} \;=\; \tag{16}$$



iii) Show that $\mathrm{CNOT}_{12}^2 = \mathbb{1} \otimes \mathbb{1}$ and that $\mathrm{CNOT}_{12}^\dagger = \mathrm{CNOT}_{12}$. Conclude that CNOT_{12} is unitary.

b) Outer-product expansion of CNOT. Argue in words why

$$\mathrm{CNOT}_{12} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X. \tag{17}$$

Expand this into a sum of two-qubit outer products $|xy\rangle\langle x'y'|$ and verify that it matches your answer in the previous subproblem.

c) Decomposing CNOT using CZ.

The controlled-Z gate is defined by

$$\mathrm{CZ} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes Z. \tag{18}$$

i) Write down the matrix for CZ in the computational basis and explain briefly why CZ is symmetric in the two qubits, i.e. it has no distinguished control and target. Since it is symmetric, we depict it in circuits as

$$\mathrm{CZ} \;=\; \tag{}$$



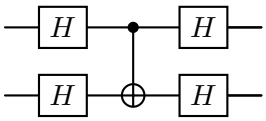
ii) Show that

$$\mathrm{CNOT}_{12} = (\mathbb{1} \otimes H)\,\mathrm{CZ}\,(\mathbb{1} \otimes H). \tag{19}$$

This means that a CNOT can be expressed as a CZ with Hadamards on the target, changing the effect from a Z-flip into an X-flip. Draw the circuit equivalence that (19) shows. Hint[6].

[5] Define basis states $|x\rangle \otimes |y\rangle$ labeled by classical bits $x, y \in \{0, 1\}$.
[6] Use $HZH = X$ and $H^2 = \mathbb{1}$.

d) Adding Hadamards. Now consider the following gate:

$$A = (H \otimes H)\,\mathrm{CNOT}_{12}\,(H \otimes H) \;=$$ 

(20)

i) Explain conceptually why there must exist some basis states for which the gate A acts exactly like CNOT_{12} acts in the computational basis. Hint[7].

ii) That is, there exists some basis $\mathcal{B}$ with orthonormal states $\{|0_\mathcal{B}\rangle, |1_\mathcal{B}\rangle\}$ such that

$$A\,|x_\mathcal{B}\rangle\,|y_\mathcal{B}\rangle = |x_\mathcal{B}\rangle\,|(x \oplus y)_\mathcal{B}\rangle, \qquad \text{for } x, y \in \{0, 1\}. \tag{21}$$

Find $\mathcal{B}$. Verify explicitly that (21) holds for these basis states.

e) How does the gate A act on the computational (Z) basis?

i) Use identity (19) from the previous part to rewrite A using only CZ and single-qubit gates. Avoid explicit 4×4 matrix multiplications. Use the symmetry of CZ to conclude that

$$A = \mathrm{CNOT}_{21}, \tag{22}$$

that is a *reversed* CNOT with control on qubit 2 and target on qubit 1. Draw the circuit equivalence that (22) shows.

ii) State the action of A on the computational basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ and write down the Boolean function on classical bits represented by these states.

f) Putting it all together. You have now learned two facts about the same gate A:

- For input/output states in the basis $\mathcal{B}$, A acts like the usual CNOT_{12}.
- For input/output states in the computational basis, A acts like CNOT_{21}.

i) Interpret in a sentence the identity $A = (H \otimes H)\,\mathrm{CNOT}_{12}\,(H \otimes H)$ as a basis-change statement between the computational basis and the basis $\mathcal{B}$. Use this together with the two learned facts above to show that when the input/output states are in the basis $\mathcal{B}$, the gate CNOT_{12} acts in reverse (!):

$$\mathrm{CNOT}_{12}\,|x_\mathcal{B}\rangle\,|y_\mathcal{B}\rangle = |(x \oplus y)_\mathcal{B}\rangle\,|y_\mathcal{B}\rangle, \qquad x, y \in \{0, 1\}. \tag{23}$$

Hint[8].

ii) Conclude in words to what degree we can say which qubit is the control and which is the target in the gate CNOT_{12}. Compare with classical circuits.

Problem 5: Hadamard Conjugation of CNOT in Qiskit (O. Leinonen)

In this problem you will run the circuit defining the operator A from the previous problem in Qiskit. A Google Colab notebook with the tasks is available here:

https://colab.research.google.com/drive/1LqTKUbJo5LHs3ntevpf3DssrO0zGQm2i

[7] Notice that $H \otimes H$ is unitary and its own inverse, and recall the equation for basis transformations of operators shown in Problem set 0: $[A]_{\mathcal{B}'} = U^\dagger\,[A]_\mathcal{B}\,U$.

[8] Use $|x_\mathcal{B}\rangle = H\,|x\rangle$, so $|x_\mathcal{B}\rangle\,|y_\mathcal{B}\rangle = (H \otimes H)\,|x\rangle\,|y\rangle$ and also $(H \otimes H)\,|x_\mathcal{B}\rangle\,|y_\mathcal{B}\rangle = |x\rangle\,|y\rangle$ since $(H \otimes H)^2 = \mathbb{1}$.