University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 2

This problem set will be discussed in the group exercise session on **Feb. 4**.

Problem 1: Two-qubit tensor products (O. Leinonen)

In this exercise you will practice tensor products of operators and states, and inner products on two qubits. We use the computational basis

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}. \tag{1}$$

a) Tensor products. Compute the matrix representations in the computational basis of

$$\mathbb{1} \otimes X, \qquad X \otimes \mathbb{1}, \qquad X \otimes Z, \tag{2}$$

where

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Is the tensor product commutative? Hint[1].

Solution: Using the block rule:

$$\mathbb{1} \otimes X = \begin{pmatrix} 1 \cdot X & 0 \cdot X \\ 0 \cdot X & 1 \cdot X \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$X \otimes \mathbb{1} = \begin{pmatrix} 0 \cdot \mathbb{1} & 1 \cdot \mathbb{1} \\ 1 \cdot \mathbb{1} & 0 \cdot \mathbb{1} \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{4}$$

$$X \otimes Z = \begin{pmatrix} 0 \cdot Z & 1 \cdot Z \\ 1 \cdot Z & 0 \cdot Z \end{pmatrix} = \begin{pmatrix} 0 & Z \\ Z & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

We see that the tensor product is *not* commutative in general: for example $\mathbb{1} \otimes X \neq X \otimes \mathbb{1}$.

b) Product states as 4-vectors. Let $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$. Compute each of the following as a linear combination of $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ and as a 4-component column vector in the computational basis:

$$|+0\rangle = |+\rangle \otimes |0\rangle, \qquad |0+\rangle = |0\rangle \otimes |+\rangle, \qquad |++\rangle = |+\rangle \otimes |+\rangle. \tag{5}$$

[1] For 2x2 matrices A and B, if $A = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$, then $A \otimes B = \begin{pmatrix} a_{00}B & a_{01}B \\ a_{10}B & a_{11}B \end{pmatrix}$.

Solution: We use linearity of the tensor product and the identifications

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}. \tag{6}$$

Using $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$,

$$\begin{aligned}
|+0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \\
|0+\rangle &= \frac{1}{\sqrt{2}}|0\rangle \otimes (|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \\
|++\rangle &= \frac{1}{2}(|0\rangle + |1\rangle) \otimes (|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}.
\end{aligned} \tag{7}$$

c) Inner products. Using either Dirac notation or your 4-vector representations above, compute

$$\langle +0|0+\rangle, \qquad \langle +0|++\rangle, \qquad \langle 0+|++\rangle. \tag{8}$$

Verify that $|+0\rangle, |0+\rangle, |++\rangle$ are normalized, and determine which pairs (if any) are orthogonal.

Solution: For the norms,

$$\langle +0|+0\rangle = \frac{1}{2}(1+1) = 1, \qquad \langle 0+|0+\rangle = \frac{1}{2}(1+1) = 1, \qquad \langle ++|++\rangle = \frac{1}{4}(1+1+1+1) = 1. \tag{9}$$

So all three states are normalized.

Using the standard inner product on $\mathbb{C}^4$ (conjugate transpose times column vector), we get

$$\langle +0|0+\rangle = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\right) = \frac{1}{2}, \tag{10}$$

and

$$\langle +0|++\rangle = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1 & 0\end{pmatrix}\right)\left(\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right) = \frac{1}{2\sqrt{2}}(1+1) = \frac{1}{\sqrt{2}}, \tag{11}$$

and similarly

$$\langle 0+|++\rangle = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1 & 0 & 0\end{pmatrix}\right)\left(\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right) = \frac{1}{2\sqrt{2}}(1+1) = \frac{1}{\sqrt{2}}. \tag{12}$$

None of the computed overlaps are 0, so none of the pairs $(|{+}0\rangle, |0{+}\rangle)$, $(|{+}0\rangle, |{+}{+}\rangle)$, $(|0{+}\rangle, |{+}{+}\rangle)$ are orthogonal.

Problem 2: Adjoint and trace of tensor products for general matrices (J. Gulla, O. Leinonen)

a) Matrix elements of the adjoint. You have seen that the adjoint of a linear operator M satisfies

$$\langle\phi|M^\dagger\psi\rangle = \langle M\phi|\psi\rangle \tag{13}$$

for all vectors $|\phi\rangle, |\psi\rangle$. Use (13) to show that for any orthonormal basis $\{|r\rangle\}$,

$$\langle r|M^\dagger|s\rangle = \langle s|M|r\rangle^*. \tag{14}$$

Solution: Take $|\phi\rangle = |r\rangle$ and $|\psi\rangle = |s\rangle$ in (13):

$$\langle r|M^\dagger|s\rangle = \langle Mr|s\rangle = \langle s|Mr\rangle^* = \langle s|M|r\rangle^*. \tag{15}$$

b) Adjoint distributes. Let $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ be square matrices. Let $\{|i\rangle\}$ be an orthonormal basis of $\mathbb{C}^m$ and $\{|j\rangle\}$ an orthonormal basis of $\mathbb{C}^n$. Define the product basis

$$|i,j\rangle = |i\rangle \otimes |j\rangle, \tag{16}$$

with the resolution of the identity

$$\mathbb{1}_m \otimes \mathbb{1}_n = \sum_i \sum_j |i,j\rangle\langle i,j|. \tag{17}$$

i) Show that

$$(A\otimes B)^\dagger = A^\dagger \otimes B^\dagger. \tag{18}$$

Hint[2].

ii) Deduce that:

- if A and B are unitary, then $A\otimes B$ is unitary;
- if A and B are Hermitian, then $A\otimes B$ is Hermitian.

Solution:

i) We compare matrix elements in the product basis $\{|i,j\rangle\}$.

First compute $\langle k,\ell|(A\otimes B)|i,j\rangle$. Using $(A\otimes B)|i,j\rangle = (A|i\rangle)\otimes(B|j\rangle)$ and inserting resolutions

[2] Compare matrix elements in the product basis and use (14) applied to $\{|i,j\rangle\}$. To evaluate $\langle k,\ell|(A\otimes B)|i,j\rangle$ you may insert resolutions of the identity on each subsystem, e.g. $\mathbb{1}_m = \sum_i |i\rangle\langle i|$ and $\mathbb{1}_n = \sum_j |j\rangle\langle j|$.

of the identity on each factor,

$$\begin{aligned}(A\otimes B)\,|i,j\rangle &= (A\,|i\rangle)\otimes(B\,|j\rangle)\\ &= \left(\sum_{k'}|k'\rangle\langle k'|\,A\,|i\rangle\right)\otimes\left(\sum_{\ell'}|\ell'\rangle\langle\ell'|\,B\,|j\rangle\right)\\ &= \sum_{k'}\sum_{\ell'}|k',\ell'\rangle\,\langle k'|A\,|i\rangle\,\langle\ell'|B\,|j\rangle.\end{aligned} \tag{19}$$

Taking the inner product with $\langle k,\ell|$ (and using orthonormality of the product basis) gives

$$\langle k,\ell|(A\otimes B)\,|i,j\rangle = \langle k|A\,|i\rangle\,\langle\ell|B\,|j\rangle. \tag{20}$$

Now apply (14) to the orthonormal product basis $\{|i,j\rangle\}$:

$$\begin{aligned}\langle i,j|(A\otimes B)^\dagger\,|k,\ell\rangle &= \langle k,\ell|(A\otimes B)\,|i,j\rangle^*\\ &= \langle k|A\,|i\rangle^*\langle\ell|B\,|j\rangle^*\\ &= \langle i|A^\dagger\,|k\rangle\,\langle j|B^\dagger\,|\ell\rangle = \langle i,j|(A^\dagger\otimes B^\dagger)\,|k,\ell\rangle.\end{aligned} \tag{21}$$

Since the matrix elements agree for all basis vectors, we conclude $(A\otimes B)^\dagger = A^\dagger\otimes B^\dagger$.

ii)

- If A, B are unitary, then part **i)** gives

$$(A\otimes B)^\dagger(A\otimes B) = (A^\dagger\otimes B^\dagger)(A\otimes B) = (A^\dagger A)\otimes(B^\dagger B) = \mathbb{1}_m\otimes\mathbb{1}_n, \tag{22}$$

  so $A\otimes B$ is unitary.

- Similarly, if A, B are Hermitian, then

$$(A\otimes B)^\dagger = A^\dagger\otimes B^\dagger = A\otimes B, \tag{23}$$

  so $A\otimes B$ is Hermitian.

c) Trace factorizes. Show that for any $A\in\mathbb{C}^{m\times m}$ and $B\in\mathbb{C}^{n\times n}$,

$$\mathrm{Tr}(A\otimes B) = \mathrm{Tr}(A)\ \mathrm{Tr}(B). \tag{24}$$

Solution:

$$\begin{aligned}\mathrm{Tr}(A\otimes B) &= \sum_{i,j}\langle i,j|(A\otimes B)\,|i,j\rangle = \sum_{i,j}\langle i|A\,|i\rangle\,\langle j|B\,|j\rangle = \left(\sum_i\langle i|A\,|i\rangle\right)\left(\sum_j\langle j|B\,|j\rangle\right)\\ &= \mathrm{Tr}(A)\ \mathrm{Tr}(B).\end{aligned} \tag{25}$$

Problem 3: Quantum gate identities (O. Leinonen)

a) The Hadamard gate is

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \tag{26}$$

Show that $H = \frac{1}{\sqrt{2}}(X+Z)$ and use this to prove that $HXH = Z$ and $HZH = X$. Hint[3].

Solution: Direct calculation gives

$$\frac{1}{\sqrt{2}}(X+Z) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = H. \tag{27}$$

For HXH, insert $H = \frac{1}{\sqrt{2}}(X+Z)$:

$$\begin{aligned} HXH &= \frac{1}{2}(X+Z)X(X+Z) \\ &= \frac{1}{2}\big(XXX + XXZ + ZXX + ZXZ\big). \end{aligned} \tag{28}$$

Now use $X^2 = \mathbb{1}$ (so $XXX = X$, $XXZ = Z$, and $ZXX = Z$), and $XZ = -ZX$ (so $ZXZ = -X$):

$$HXH = \frac{1}{2}(X + Z + Z - X) = Z. \tag{29}$$

To get $HZH = X$, note that

$$H^2 = \frac{1}{2}(X+Z)^2 = \frac{1}{2}(X^2 + Z^2 + XZ + ZX) = \frac{1}{2}(\mathbb{1} + \mathbb{1} + 0) = \mathbb{1}. \tag{30}$$

Multiplying $HXH = Z$ on the left and right by H then gives

$$H(HXH)H = HZH \quad \Rightarrow \quad X = HZH. \tag{31}$$

b) The S and T gates are:

$$S = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}, \quad T = \begin{pmatrix}1 & 0\\ 0 & e^{i\pi/4}\end{pmatrix}. \tag{32}$$

i) Show that $T^2 = S$, $S^2 = Z$, and therefore $T^8 = \mathbb{1}$.

ii) Show that

$$S = e^{i\pi/4}\, e^{-i\pi Z/4}, \tag{33}$$

meaning that S is a rotation about the Z-axis (up to global phase).

iii) Let $Y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = iXZ$ and show that $SXS^\dagger = Y$ and $SZS^\dagger = Z$. Hint[4].

Solution:

i)

$$T^2 = \begin{pmatrix}1 & 0\\ 0 & e^{i\pi/2}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} = S, \qquad S^2 = \begin{pmatrix}1 & 0\\ 0 & i^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = Z, \tag{34}$$

so

$$T^8 = (T^2)^4 = S^4 = (S^2)^2 = Z^2 = \mathbb{1}. \tag{35}$$

[3] Use $X^2 = Z^2 = \mathbb{1}$ and $XZ = -ZX$.

[4] For the rotation, recall that $e^{-i\frac{\theta}{2}Z} = \cos\frac{\theta}{2}\mathbb{1} - i\sin\frac{\theta}{2}Z$.

ii) For the rotation:

$$e^{-i\pi Z/4} = \cos(\pi/4)\,\mathbb{1} - i\sin(\pi/4)\,Z = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}. \tag{36}$$

Therefore

$$e^{i\pi/4}e^{-i\pi Z/4} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = S. \tag{37}$$

(The prefactor $e^{i\pi/4}$ is a global phase.)

iii) Finally, $S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$, so

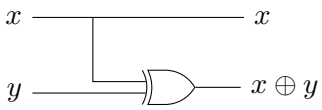
$$SXS^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = Y. \tag{38}$$

Also S is diagonal, hence it commutes with Z, and therefore $SZS^\dagger = Z$.

Problem 4: Control and target qubits in CNOT (J. Gulla, O. Leinonen)

The goal of this problem is to understand and be able to predict what happens in circuits containing CNOT gates. Throughout, CNOT_{12} means a CNOT with control on qubit 1 and target on qubit 2.

a) Implementing a classical controlled-XOR. Consider the following classical circuit on bits x and y:



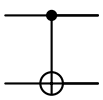
This implements an XOR on y controlled by x, while keeping x unchanged:

$$(x, y) \mapsto (x, x \oplus y). \tag{39}$$

Let us see what happens when we implement such a circuit on a quantum computer.

i) Treating the computational basis labels as classical bits $x, y \in \{0, 1\}$, translate the action of (39) as a basis state update map. Hint[5].

ii) Use your answer to write down a 4×4 matrix representing this quantum gate in the computational basis. We call this the CNOT_{12} gate, where the control (qubit 1) is unchanged and the target (qubit 2) flips if and only if qubit 1 is $|1\rangle$. It has the quantum circuit representation:

CNOT_{12} =  (40)

iii) Show that $\mathrm{CNOT}_{12}^2 = \mathbb{1} \otimes \mathbb{1}$ and that $\mathrm{CNOT}_{12}^\dagger = \mathrm{CNOT}_{12}$. Conclude that CNOT_{12} is unitary.

[5] Define basis states $|x\rangle \otimes |y\rangle$ labeled by classical bits $x, y \in \{0, 1\}$.

Solution:

i) Let the computational basis states $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ be $|x\rangle\,|y\rangle$ labeled by classical bits $x, y \in \{0, 1\}$. The Boolean function (39) is the basis map:

$$|x\rangle\,|y\rangle \;\mapsto\; |x\rangle\,|x \oplus y\rangle. \tag{41}$$

In particular, on the computational basis states:

$$|00\rangle \mapsto |00\rangle, \quad |01\rangle \mapsto |01\rangle, \quad |10\rangle \mapsto |11\rangle, \quad |11\rangle \mapsto |10\rangle. \tag{42}$$

ii) The i-th column of the matrix for this gate is the vector of the output on the i-th basis input. Hence

$$\mathrm{CNOT}_{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{43}$$

iii) To show $\mathrm{CNOT}_{12}^2 = \mathbb{1} \otimes \mathbb{1}$, use the basis-state rule:

$$\mathrm{CNOT}_{12} : \; |x\rangle\,|y\rangle \mapsto |x\rangle\,|x \oplus y\rangle. \tag{44}$$

Applying it twice gives $|x\rangle\,|y\rangle \mapsto |x\rangle\,|x \oplus (x \oplus y)\rangle = |x\rangle\,|y\rangle$, since $x \oplus x = 0$. Thus $\mathrm{CNOT}_{12}^2 = \mathbb{1} \otimes \mathbb{1}$. We also see directly from the matrix form that $\mathrm{CNOT}_{12}^\dagger = \mathrm{CNOT}_{12}$. Hence $\mathrm{CNOT}_{12}^\dagger\,\mathrm{CNOT}_{12} = \mathbb{1}$, meaning that CNOT_{12} is unitary.

b) Outer-product expansion of CNOT. Argue in words why

$$\mathrm{CNOT}_{12} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X. \tag{45}$$

Expand this into a sum of two-qubit outer products $|xy\rangle\langle x'y'|$ and verify that it matches your answer in the previous subproblem.

Solution: The operator $|0\rangle\langle 0|$ projects the control qubit onto $|0\rangle$, and $|1\rangle\langle 1|$ projects it onto $|1\rangle$. Thus (45) says: "if the control is $|0\rangle$, do $\mathbb{1}$ on the target; if the control is $|1\rangle$, do X on the target," which is exactly the CNOT rule.

Formally, expand $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$ and $X = |0\rangle\langle 1| + |1\rangle\langle 0|$ and write:

$$|0\rangle\langle 0| \otimes \mathbb{1} = |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) = |00\rangle\langle 00| + |01\rangle\langle 01|, \tag{46}$$

and

$$|1\rangle\langle 1| \otimes X = |1\rangle\langle 1| \otimes (|0\rangle\langle 1| + |1\rangle\langle 0|) = |10\rangle\langle 11| + |11\rangle\langle 10|. \tag{47}$$

Therefore

$$|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 11| + |11\rangle\langle 10| = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{48}$$

which is exactly the matrix for CNOT_{12} found earlier.

c) Decomposing CNOT using CZ.

The controlled-Z gate is defined by

$$\mathrm{CZ} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes Z. \tag{49}$$

i) Write down the matrix for CZ in the computational basis and explain briefly why CZ is symmetric in the two qubits, i.e. it has no distinguished control and target. Since it is symmetric, we depict it in circuits as

CZ =



ii) Show that

$$\mathrm{CNOT}_{12} = (\mathbb{1} \otimes H)\,\mathrm{CZ}\,(\mathbb{1} \otimes H). \tag{50}$$

This means that a CNOT can be expressed as a CZ with Hadamards on the target, changing the effect from a Z-flip into an X-flip. Draw the circuit equivalence that (50) shows. Hint[6].

Solution:

i) In the computational basis,

$$\mathrm{CZ} = \mathrm{diag}(1, 1, 1, -1), \tag{51}$$

so it acts as

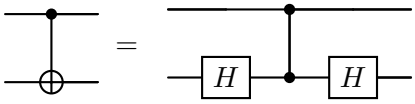
$$|00\rangle \mapsto |00\rangle, \quad |01\rangle \mapsto |01\rangle, \quad |10\rangle \mapsto |10\rangle, \quad |11\rangle \mapsto -|11\rangle. \tag{52}$$

This action is unchanged if the two qubits are swapped: the only special case is when *both* qubits are $|1\rangle$. Hence CZ is symmetric (it has no distinguished control and target).

ii) Using $HZH = X$ and $H\mathbb{1}H = \mathbb{1}$,

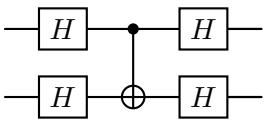
$$\begin{aligned}(\mathbb{1} \otimes H)\,\mathrm{CZ}\,(\mathbb{1} \otimes H) &= (\mathbb{1} \otimes H)\left(|0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes Z\right)(\mathbb{1} \otimes H) \\ &= |0\rangle\langle 0| \otimes (H\mathbb{1}H) + |1\rangle\langle 1| \otimes (HZH) \\ &= |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes X \\ &= \mathrm{CNOT}_{12}.\end{aligned} \tag{53}$$

In circuit diagrams:



d) Adding Hadamards. Now consider the following gate:

$$A = (H \otimes H)\,\mathrm{CNOT}_{12}\,(H \otimes H) \;=$$



(54)

i) Explain conceptually why there must exist some basis states for which the gate A acts exactly like CNOT_{12} acts in the computational basis. Hint[7].

[6] Use $HZH = X$ and $H^2 = \mathbb{1}$.

[7] Notice that $H \otimes H$ is unitary and its own inverse, and recall the equation for basis transformations of operators shown in Problem set 0: $[A]_{\mathcal{B}'} = U^\dagger\,[A]_{\mathcal{B}}\,U$.

ii) That is, there exists some basis $\mathcal{B}$ with orthonormal states $\{|0_\mathcal{B}\rangle, |1_\mathcal{B}\rangle\}$ such that

$$A\,|x_\mathcal{B}\rangle\,|y_\mathcal{B}\rangle = |x_\mathcal{B}\rangle\,|(x\oplus y)_\mathcal{B}\rangle, \qquad \text{for } x,y\in\{0,1\}. \tag{55}$$

Find $\mathcal{B}$. Verify explicitly that (55) holds for these basis states.

Solution:

i) Since H is unitary and $H^\dagger = H$, applying $H\otimes H$ on both sides corresponds to expressing the same underlying operation in a rotated basis. Concretely, the circuit for A can be read as: change basis on both qubits using $H\otimes H$, apply the usual CNOT, then change basis back. Therefore one expects a clean CNOT-type truth table when inputs and outputs are described in the basis associated with the Hadamard rotation.

ii) Choose the basis to be the Hadamard rotations of the computational basis:

$$|0_\mathcal{B}\rangle = H\,|0\rangle = |+\rangle, \qquad |1_\mathcal{B}\rangle = H\,|1\rangle = |-\rangle, \qquad |\pm\rangle = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}. \tag{56}$$

(This is the X-eigenbasis.) Then

$$|x_\mathcal{B}\rangle\,|y_\mathcal{B}\rangle = (H\otimes H)\,|x\rangle\,|y\rangle, \qquad (H\otimes H)\,|x_\mathcal{B}\rangle\,|y_\mathcal{B}\rangle = |x\rangle\,|y\rangle, \tag{57}$$

since $(H\otimes H)^2 = \mathbb{1}$. Hence

$$\begin{aligned} A\,|x_\mathcal{B}\rangle\,|y_\mathcal{B}\rangle &= (H\otimes H)\,\mathrm{CNOT}_{12}\,(H\otimes H)\,|x_\mathcal{B}\rangle\,|y_\mathcal{B}\rangle \\ &= (H\otimes H)\,\mathrm{CNOT}_{12}\,|x\rangle\,|y\rangle \\ &= (H\otimes H)\,|x\rangle\,|x\oplus y\rangle \\ &= |x_\mathcal{B}\rangle\,|(x\oplus y)_\mathcal{B}\rangle. \end{aligned}$$

e) How does the gate A act on the computational (Z) basis?

i) Use identity (50) from the previous part to rewrite A using only CZ and single-qubit gates. Avoid explicit 4×4 matrix multiplications. Use the symmetry of CZ to conclude that

$$A = \mathrm{CNOT}_{21}, \tag{58}$$

that is a *reversed* CNOT with control on qubit 2 and target on qubit 1. Draw the circuit equivalence that (58) shows.

ii) State the action of A on the computational basis states $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ and write down the Boolean function on classical bits represented by these states.

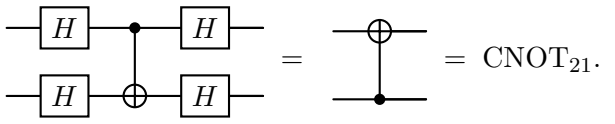
Solution:

i) Substitute $\mathrm{CNOT}_{12} = (\mathbb{1}\otimes H)\,\mathrm{CZ}\,(\mathbb{1}\otimes H)$:

$$\begin{aligned} A &= (H\otimes H)\,\mathrm{CNOT}_{12}\,(H\otimes H) \\ &= (H\otimes H)\,(\mathbb{1}\otimes H)\,\mathrm{CZ}\,(\mathbb{1}\otimes H)\,(H\otimes H) \\ &= (H\otimes \mathbb{1})\,\mathrm{CZ}\,(H\otimes \mathbb{1}). \end{aligned}$$

Since CZ is symmetric, we may equally view it as applying a Z on qubit 1 conditional on qubit 2 being $|1\rangle$. Conjugating that conditional Z on qubit 1 by H turns it into a conditional X on qubit 1. Thus $(H\otimes\mathbb{1})\,\mathrm{CZ}\,(H\otimes\mathbb{1})$ is a CNOT with target on qubit 1 and control on qubit 2, i.e.

CNOT_{21}. Therefore $A = \mathrm{CNOT}_{21}$. In circuit diagrams:



ii) The gate $A = \mathrm{CNOT}_{21}$ keeps the control (qubit 2) unchanged and flips the target (qubit 1) if and only if qubit 2 is $|1\rangle$:

$$|00\rangle \mapsto |00\rangle, \quad |01\rangle \mapsto |11\rangle, \quad |10\rangle \mapsto |10\rangle, \quad |11\rangle \mapsto |01\rangle. \tag{59}$$

Letting these states be labeled $|x\rangle\,|y\rangle$ for $x, y \in \{0, 1\}$, the CNOT_{21} induces the Boolean function

$$\mathrm{CNOT}_{21}\,|x\rangle\,|y\rangle = |x \oplus y\rangle\,|y\rangle. \tag{60}$$

f) Putting it all together. You have now learned two facts about the same gate A:

- For input/output states in the basis $\mathcal{B}$, A acts like the usual CNOT_{12}.
- For input/output states in the computational basis, A acts like CNOT_{21}.

i) Interpret in a sentence the identity $A = (H \otimes H)\,\mathrm{CNOT}_{12}\,(H \otimes H)$ as a basis-change statement between the computational basis and the basis $\mathcal{B}$. Use this together with the two learned facts above to show that when the input/output states are in the basis $\mathcal{B}$, the gate CNOT_{12} acts in reverse (!):

$$\mathrm{CNOT}_{12}\,|x_{\mathcal{B}}\rangle\,|y_{\mathcal{B}}\rangle = |(x \oplus y)_{\mathcal{B}}\rangle\,|y_{\mathcal{B}}\rangle, \qquad x, y \in \{0, 1\}. \tag{61}$$

Hint[8].

ii) Conclude in words to what degree we can say which qubit is the control and which is the target in the gate CNOT_{12}. Compare with classical circuits.

Solution:

i) Recall $|x_{\mathcal{B}}\rangle = H\,|x\rangle$. Thus $H \otimes H$ is exactly the change of logical basis between computational labels and $\mathcal{B}$-labels (to and from):

$$|x_{\mathcal{B}}\rangle\,|y_{\mathcal{B}}\rangle = (H \otimes H)\,|x\rangle\,|y\rangle, \qquad (H \otimes H)\,|x_{\mathcal{B}}\rangle\,|y_{\mathcal{B}}\rangle = |x\rangle\,|y\rangle. \tag{62}$$

In this language, the conjugation

$$A = (H \otimes H)\,\mathrm{CNOT}_{12}\,(H \otimes H) \tag{63}$$

says: *A is what CNOT_{12} looks like when the logical bits are encoded in the basis $\mathcal{B}$, but then written as an operator in the computational basis.* From the previous part we also know that, in the computational basis, $A = \mathrm{CNOT}_{21}$. Hence

$$\mathrm{CNOT}_{12} = (H \otimes H)\,A\,(H \otimes H) = (H \otimes H)\,\mathrm{CNOT}_{21}\,(H \otimes H). \tag{64}$$

[8] Use $|x_{\mathcal{B}}\rangle = H\,|x\rangle$, so $|x_{\mathcal{B}}\rangle\,|y_{\mathcal{B}}\rangle = (H \otimes H)\,|x\rangle\,|y\rangle$ and also $(H \otimes H)\,|x_{\mathcal{B}}\rangle\,|y_{\mathcal{B}}\rangle = |x\rangle\,|y\rangle$ since $(H \otimes H)^2 = \mathbb{1}$.

Apply this identity to $|x_{\mathcal{B}}\rangle\,|y_{\mathcal{B}}\rangle$:

$$\begin{aligned}\mathrm{CNOT}_{12}\,|x_{\mathcal{B}}\rangle\,|y_{\mathcal{B}}\rangle &= (H\otimes H)\,\mathrm{CNOT}_{21}\,(H\otimes H)\,|x_{\mathcal{B}}\rangle\,|y_{\mathcal{B}}\rangle\\ &= (H\otimes H)\,\mathrm{CNOT}_{21}\,|x\rangle\,|y\rangle\\ &= (H\otimes H)\,|x\oplus y\rangle\,|y\rangle\\ &= |(x\oplus y)_{\mathcal{B}}\rangle\,|y_{\mathcal{B}}\rangle,\end{aligned}$$

which is exactly the truth table of a reversed CNOT on the $\mathcal{B}$-encoded bits.

ii) In a classical circuit for $(x, y) \mapsto (x, x \oplus y)$, "control" and "target" are fixed by the wiring: the control wire is fed into the XOR, and the target wire is the one updated. For a quantum unitary, "control" and "target" refer to a chosen logical 0/1 encoding. In the computational (Z) basis, CNOT_{12} has control on qubit 1 and target on qubit 2, but in the X-eigenbasis $\mathcal{B} = \{|+\rangle, |-\rangle\}$ the *same* unitary implements the reversed truth table. Thus, unlike the classical case, control/target can swap under local basis changes.

Problem 5: Hadamard Conjugation of CNOT in Qiskit (O. Leinonen)

In this problem you will run the circuit defining the operator A from the previous problem in Qiskit. A Google Colab notebook with the tasks is available here:

https://colab.research.google.com/drive/1LqTKUbJo5LHs3ntevpf3DssrO0zGQm2i

Solution: Solutions to the tasks are available here:

https://colab.research.google.com/drive/1nUF-jKFT4KGUNLON2mGRs0JBC5wcxGfT