University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 3

This problem set will be discussed on **Feb. 11**.

Problem 1: Entanglement and Bell States (O. Leinonen)

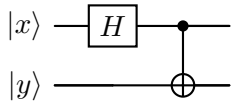
The four Bell states are maximally entangled two-qubit states, defined as:

$$\left|\Phi^+\right\rangle = |\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad \left|\Phi^-\right\rangle = |\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$
$$\left|\Psi^+\right\rangle = |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \qquad \left|\Psi^-\right\rangle = |\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

The Controlled-NOT (CNOT) gate is defined by the matrix:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1}$$

a) Show that the following circuit produces the four Bell states $|\beta_{xy}\rangle$ from the basis states $|xy\rangle$ (where $x, y \in \{0, 1\}$):



b) A state is called a *product state* if it can be written as $|\psi\rangle = |a\rangle \otimes |b\rangle$. Otherwise, it is *entangled*. Show that the Bell state $|\Phi^+\rangle$ is entangled. Hint[1].

c) It is possible to transform the Bell state $|\Phi^+\rangle$ into any of the other three Bell states by applying a gate to **only the first qubit**. Show that applying the Pauli gates σ_x, σ_z, and σ_y to the first qubit of $|\Phi^+\rangle$ produces the states $|\Psi^+\rangle$, $|\Phi^-\rangle$, and $|\Psi^-\rangle$ respectively (up to a global phase factor).

d) Consider the Bell state $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$. We measure the observable $M = \sigma_z \otimes \sigma_z$ on the joint system.

- **i)** Show that $|\Phi^+\rangle$ is an eigenstate of $\sigma_z \otimes \sigma_z$. What is the eigenvalue?
- **ii)** Based on **i)**, what is the expectation value $\langle \sigma_z \otimes \sigma_z \rangle$? Explain the physical meaning of this result in terms of correlations between qubit A and qubit B.

Problem 2: Projective Measurements and the Born Rule (O. Leinonen)

A projective measurement is described by a set of orthogonal projectors $\{P_m\}$ satisfying:

1. **Completeness:** $\sum_m P_m = \mathbb{1}$
2. **Orthogonality:** $P_m P_{m'} = \delta_{mm'} P_m$

[1] Write down the coefficient matrix and calculate its rank.

For a system in state $|\psi\rangle$, the probability of outcome m is given by the *Born rule*:

$$p(m) = \langle\psi|P_m|\psi\rangle \tag{2}$$

and the post-measurement state is:

$$|\psi_{\text{post}}\rangle = \frac{P_m|\psi\rangle}{\sqrt{p(m)}} \tag{3}$$

a) Consider measurement in the computational basis with projectors $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$.

i) Verify that the completeness relation $P_0 + P_1 = \mathbb{1}$ holds.

ii) Show that the orthogonality condition $P_m P_{m'} = \delta_{mm'} P_m$ is satisfied for all $m, m' \in \{0, 1\}$.

b) Starting with the state $|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$, consider performing *two* measurements in sequence: first in the computational basis $\{|0\rangle, |1\rangle\}$, then immediately in the Hadamard basis $\{|+\rangle, |-\rangle\}$.

i) If the first measurement gives outcome 0, what are the probabilities for the second measurement?

ii) If the first measurement gives outcome 1, what are the probabilities for the second measurement?

iii) Calculate the joint probability $p(0, +)$ of getting outcome 0 in the first measurement AND outcome + in the second. Do the same for all four joint outcomes.

Hint[2].

c) Define the operators $N_{ij} = P_j^{(\pm)} P_i^{(01)}$ for $i \in \{0, 1\}$ and $j \in \{+, -\}$, where $P_0^{(01)} = |0\rangle\langle 0|$, $P_1^{(01)} = |1\rangle\langle 1|$, $P_+^{(\pm)} = |+\rangle\langle +|$, $P_-^{(\pm)} = |-\rangle\langle -|$.

i) Show that $p(i, j) = \langle\psi|N_{ij}^\dagger N_{ij}|\psi\rangle$ gives the same joint probabilities you calculated above.

ii) Verify that the four operators $\{N_{ij}\}$ satisfy the completeness relation $\sum_{i,j} N_{ij}^\dagger N_{ij} = \mathbb{1}$.

This shows the cascaded measurement is equivalent to a single measurement with operators $\{N_{ij}\}$.

d) The previous parts showed that a specific cascaded measurement can be described by a single set of measurement operators. Now prove this in general.

Suppose $\{L_\ell\}$ and $\{M_m\}$ are two sets of measurement operators (not necessarily projective). Show that performing measurement $\{L_\ell\}$ followed by measurement $\{M_m\}$ is equivalent to a single measurement with operators $\{N_{\ell m}\}$ where $N_{\ell m} = M_m L_\ell$.

Verify that the $\{N_{\ell m}\}$ satisfy the completeness relation $\sum_{\ell,m} N_{\ell m}^\dagger N_{\ell m} = \mathbb{1}$. Hint[3].

[2] Use $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$.

[3] Use the completeness relations for $\{L_\ell\}$ and $\{M_m\}$.

Problem 3: Observables, Expectation Values, and the Uncertainty Principle (O. Leinonen)

An *observable* is a Hermitian operator M with spectral decomposition:

$$M = \sum_m m\, P_m \tag{4}$$

where m are the eigenvalues (measurement outcomes) and P_m are the projectors onto the corresponding eigenspaces. The expectation value of M in state $|\psi\rangle$ is:

$$\langle M \rangle = \langle \psi | M | \psi \rangle = \sum_m m\, p(m) \tag{5}$$

where $p(m) = \langle \psi | P_m | \psi \rangle$ is the probability of outcome m.

a) Consider the observable σ_z with eigenvalues ± 1.

i) Write the spectral decomposition of σ_z in terms of projectors.

ii) Show that the projectors can be written as $P_\pm = \frac{1}{2}(\mathbb{1} \pm \sigma_z)$.

b) For the general qubit state $|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$:

i) Calculate $\langle \sigma_z \rangle$ using the probability formula $\langle \sigma_z \rangle = (+1)p(+) + (-1)p(-)$.

ii) Calculate $\langle \sigma_z \rangle$ directly using $\langle \sigma_z \rangle = \langle \psi | \sigma_z | \psi \rangle$ and verify the results agree.

c) The *variance* of an observable M is defined as:

$$(\Delta M)^2 = \langle M^2 \rangle - \langle M \rangle^2 \tag{6}$$

and the *standard deviation* (or uncertainty) is $\Delta M = \sqrt{(\Delta M)^2}$.

Calculate $\Delta\sigma_z$ for $|\psi(\theta,\phi)\rangle$ and interpret the result geometrically on the Bloch sphere. Hint[4].

d) (Heisenberg Uncertainty Principle)

For any two observables A and B, the uncertainty principle states:

$$\Delta A \cdot \Delta B \geq \frac{1}{2}\left|\langle [A,B] \rangle\right| \tag{7}$$

where $[A,B] = AB - BA$ is the commutator.

Consider $A = \sigma_x$ and $B = \sigma_y$ for the state $|0\rangle$.

i) Using $[\sigma_x, \sigma_y] = 2i\sigma_z$, evaluate the right-hand side of the uncertainty relation.

ii) Calculate $\Delta\sigma_x$ and $\Delta\sigma_y$ for $|0\rangle$ and verify the uncertainty relation is satisfied.

Hint[5].

e) Find a state for which the uncertainty product $\Delta\sigma_x \cdot \Delta\sigma_y$ is strictly greater than the lower bound. How does the position of this state on the Bloch sphere compare to $|0\rangle$?

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[4] Note that $\sigma_z^2 = \mathbb{1}$.
[5] Use $\sigma_x^2 = \sigma_y^2 = \mathbb{1}$.

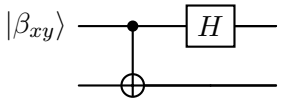
Problem 4: Bell Measurement and Superdense Coding (J. Gulla)

A *Bell measurement* is a projective measurement in the Bell basis $\{|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle\}$, where

$$|\beta_{00}\rangle = \left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\beta_{10}\rangle = \left|\Phi^-\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$
$$|\beta_{01}\rangle = \left|\Psi^+\right\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\beta_{11}\rangle = \left|\Psi^-\right\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

In the previous exercise you showed that the circuit $U_{\text{prep}} = \text{CNOT}(H \otimes \mathbb{1})$ maps $|xy\rangle \mapsto |\beta_{xy}\rangle$.

a) (**Bell decoding**) Show that the inverse circuit maps Bell states back to computational basis states:



i.e., $\quad |\beta_{xy}\rangle \mapsto |xy\rangle.$

b) (**Bell measurement**) Explain why applying the decoding circuit above and then measuring both qubits in the computational basis implements a Bell measurement, i.e. measures in the Bell basis.

c) (**Superdense coding**) Alice and Bob share the entangled state $|\Phi^+\rangle = |\beta_{00}\rangle$. Alice wants to send two classical bits $(x, y) \in \{0, 1\}^2$ to Bob by sending *one* qubit.

- **i)** Show that if Alice applies the single-qubit operation $\sigma_z^x \sigma_x^y$ to *her* qubit, the joint state becomes $|\beta_{xy}\rangle$ (up to a global phase).
- **ii)** Explain how Bob can recover (x, y) with certainty using a Bell measurement, and draw the decoding circuit.