University of Oslo

FYS3415/FYS4415 — Quantum computing and quantum information

Problem set 3

This problem set will be discussed on **Feb. 11**.

Problem 1: Entanglement and Bell States (O. Leinonen)

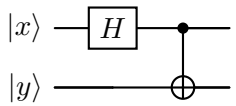
The four Bell states are maximally entangled two-qubit states, defined as:

$$\left|\Phi^+\right\rangle = |\beta_{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad \left|\Phi^-\right\rangle = |\beta_{10}\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$
$$\left|\Psi^+\right\rangle = |\beta_{01}\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \qquad \left|\Psi^-\right\rangle = |\beta_{11}\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

The Controlled-NOT (CNOT) gate is defined by the matrix:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{1}$$

a) Show that the following circuit produces the four Bell states $|\beta_{xy}\rangle$ from the basis states $|xy\rangle$ (where $x, y \in \{0, 1\}$):



Solution: For the input $|00\rangle$:

$$|00\rangle \xrightarrow{H\otimes\mathbb{1}} \frac{(|0\rangle + |1\rangle)\,|0\rangle}{\sqrt{2}} = \frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow{\text{CNOT}} \frac{|00\rangle + |11\rangle}{\sqrt{2}} = |\beta_{00}\rangle \quad (\text{or } \left|\Phi^+\right\rangle) \tag{2}$$

For the input $|01\rangle$:

$$|01\rangle \xrightarrow{H\otimes\mathbb{1}} \frac{(|0\rangle + |1\rangle)\,|1\rangle}{\sqrt{2}} = \frac{|01\rangle + |11\rangle}{\sqrt{2}} \xrightarrow{\text{CNOT}} \frac{|01\rangle + |10\rangle}{\sqrt{2}} = |\beta_{01}\rangle \quad (\text{or } \left|\Psi^+\right\rangle) \tag{3}$$

(Similarly, $|10\rangle \to |\Phi^-\rangle$ and $|11\rangle \to |\Psi^-\rangle$.)

b) A state is called a *product state* if it can be written as $|\psi\rangle = |a\rangle \otimes |b\rangle$. Otherwise, it is *entangled*. Show that the Bell state $|\Phi^+\rangle$ is entangled. Hint[1].

Solution: Write $|\Phi^+\rangle = \sum_{i,j\in\{0,1\}} c_{ij}\,|ij\rangle$ and collect the coefficients into

$$C(\Phi^+) = \begin{pmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

[1] Write down the coefficient matrix and calculate its rank.

Since $\det C(\Phi^+) = \frac{1}{2} \neq 0$, we have $\operatorname{rank} C(\Phi^+) = 2$. A two-qubit pure state is a product state iff its coefficient matrix has rank 1, so $|\Phi^+\rangle$ is not a product state. Hence it is entangled.

c) It is possible to transform the Bell state $|\Phi^+\rangle$ into any of the other three Bell states by applying a gate to **only the first qubit**. Show that applying the Pauli gates σ_x, σ_z, and σ_y to the first qubit of $|\Phi^+\rangle$ produces the states $|\Psi^+\rangle$, $|\Phi^-\rangle$, and $|\Psi^-\rangle$ respectively (up to a global phase factor).

Solution: We start with $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$.

1. Applying σ_x to the first qubit:

$$(\sigma_x \otimes \mathbb{1})\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|10\rangle + |01\rangle}{\sqrt{2}} = \left|\Psi^+\right\rangle \quad (\text{or } |\beta_{01}\rangle) \tag{4}$$

2. Applying σ_z to the first qubit:

$$(\sigma_z \otimes \mathbb{1})\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \left|\Phi^-\right\rangle \quad (\text{or } |\beta_{10}\rangle) \tag{5}$$

3. Applying σ_y to the first qubit (recall $\sigma_y|0\rangle = i|1\rangle$ and $\sigma_y|1\rangle = -i|0\rangle$):

$$(\sigma_y \otimes \mathbb{1})\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{i|10\rangle - i|01\rangle}{\sqrt{2}} = -i\left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right) = -i\left|\Psi^-\right\rangle \quad (\text{or } -i|\beta_{11}\rangle) \tag{6}$$

d) Consider the Bell state $|\Phi^+\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$. We measure the observable $M = \sigma_z \otimes \sigma_z$ on the joint system.

i) Show that $|\Phi^+\rangle$ is an eigenstate of $\sigma_z \otimes \sigma_z$. What is the eigenvalue?

ii) Based on **i)**, what is the expectation value $\langle \sigma_z \otimes \sigma_z \rangle$? Explain the physical meaning of this result in terms of correlations between qubit A and qubit B.

Solution:

i) Apply the operator to the state:

$$(\sigma_z \otimes \sigma_z)\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}[(\sigma_z|0\rangle \otimes \sigma_z|0\rangle) + (\sigma_z|1\rangle \otimes \sigma_z|1\rangle)] \tag{7}$$

Using $\sigma_z|0\rangle = (+1)|0\rangle$ and $\sigma_z|1\rangle = (-1)|1\rangle$:

$$= \frac{1}{\sqrt{2}}[((+1)(+1)|00\rangle) + ((-1)(-1)|11\rangle)] = \frac{|00\rangle + |11\rangle}{\sqrt{2}} = 1 \cdot \left|\Phi^+\right\rangle \tag{8}$$

Thus, $|\Phi^+\rangle$ is an eigenstate with eigenvalue $\lambda = +1$.

ii) Since the state is an eigenstate with eigenvalue $+1$, the expectation value is exactly $\langle \sigma_z \otimes \sigma_z \rangle = 1$.

Physical meaning: The measurement outcome of $\sigma_z \otimes \sigma_z$ is the product of the individual outcomes ($Z_A \cdot Z_B$). An expectation value of $+1$ means the product is always positive. This implies that if we measure qubit A and get $+1$, qubit B must be $+1$. If A is -1, B must be

−1. The measurement results are perfectly correlated. Note that each qubit *individually* is still random in the Z-basis: $\langle \sigma_z \otimes \mathbb{1} \rangle = \langle \mathbb{1} \otimes \sigma_z \rangle = 0$ for $|\Phi^+\rangle$, so each local outcome ± 1 occurs with probability 1/2.

Problem 2: Projective Measurements and the Born Rule (O. Leinonen)

A projective measurement is described by a set of orthogonal projectors $\{P_m\}$ satisfying:

1. **Completeness:** $\sum_m P_m = \mathbb{1}$
2. **Orthogonality:** $P_m P_{m'} = \delta_{mm'} P_m$

For a system in state $|\psi\rangle$, the probability of outcome m is given by the *Born rule*:

$$p(m) = \langle \psi | P_m | \psi \rangle \tag{9}$$

and the post-measurement state is:

$$|\psi_{\text{post}}\rangle = \frac{P_m |\psi\rangle}{\sqrt{p(m)}} \tag{10}$$

a) Consider measurement in the computational basis with projectors $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$.

i) Verify that the completeness relation $P_0 + P_1 = \mathbb{1}$ holds.

ii) Show that the orthogonality condition $P_m P_{m'} = \delta_{mm'} P_m$ is satisfied for all $m, m' \in \{0, 1\}$.

Solution:

i) Using $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$:

$$P_0 + P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1} \tag{11}$$

ii) Check all combinations:

$$\begin{aligned} P_0 P_0 &= |0\rangle\langle 0|0\rangle\langle 0| = |0\rangle\langle 0| = P_0 \quad \checkmark \\ P_1 P_1 &= |1\rangle\langle 1|1\rangle\langle 1| = |1\rangle\langle 1| = P_1 \quad \checkmark \\ P_0 P_1 &= |0\rangle\langle 0|1\rangle\langle 1| = 0 \quad \checkmark \\ P_1 P_0 &= |1\rangle\langle 1|0\rangle\langle 0| = 0 \quad \checkmark \end{aligned}$$

Thus $P_m P_{m'} = \delta_{mm'} P_m$ is satisfied for all m, m'.

b) Starting with the state $|\psi\rangle = \frac{1}{\sqrt{3}} |0\rangle + \sqrt{\frac{2}{3}} |1\rangle$, consider performing *two* measurements in sequence: first in the computational basis $\{|0\rangle, |1\rangle\}$, then immediately in the Hadamard basis $\{|+\rangle, |-\rangle\}$.

i) If the first measurement gives outcome 0, what are the probabilities for the second measurement?

ii) If the first measurement gives outcome 1, what are the probabilities for the second measurement?

iii) Calculate the joint probability $p(0,+)$ of getting outcome 0 in the first measurement AND outcome + in the second. Do the same for all four joint outcomes.

Hint[2].

Solution:

i) If the first measurement gives outcome 0, the post-measurement state is $|0\rangle$.

Express in Hadamard basis: $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$.

Probabilities for second measurement: $p(+|0) = p(-|0) = \frac{1}{2}$.

ii) If outcome 1, post-measurement state is $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$.

Probabilities: $p(+|1) = p(-|1) = \frac{1}{2}$.

iii) Joint probabilities using $p(i,j) = p(i) \cdot p(j|i)$:

$$p(0,+) = p(0) \cdot p(+|0) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$
$$p(0,-) = p(0) \cdot p(-|0) = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$
$$p(1,+) = p(1) \cdot p(+|1) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$
$$p(1,-) = p(1) \cdot p(-|1) = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$$

Note: $\sum_{i,j} p(i,j) = \frac{1}{6} + \frac{1}{6} + \frac{1}{3} + \frac{1}{3} = 1$ ✓

c) Define the operators $N_{ij} = P_j^{(\pm)} P_i^{(01)}$ for $i \in \{0,1\}$ and $j \in \{+,-\}$, where $P_0^{(01)} = |0\rangle\langle 0|$, $P_1^{(01)} = |1\rangle\langle 1|$, $P_+^{(\pm)} = |+\rangle\langle +|$, $P_-^{(\pm)} = |-\rangle\langle -|$.

i) Show that $p(i,j) = \langle\psi| N_{ij}^\dagger N_{ij} |\psi\rangle$ gives the same joint probabilities you calculated above.

ii) Verify that the four operators $\{N_{ij}\}$ satisfy the completeness relation $\sum_{i,j} N_{ij}^\dagger N_{ij} = \mathbb{1}$.

This shows the cascaded measurement is equivalent to a single measurement with operators $\{N_{ij}\}$.

Solution:

i) Compute $N_{ij}^\dagger N_{ij}$. Since projectors satisfy $P^\dagger = P$ and $P^2 = P$:

$$N_{ij}^\dagger N_{ij} = P_i^{(01)} P_j^{(\pm)} P_j^{(\pm)} P_i^{(01)} = P_i^{(01)} P_j^{(\pm)} P_i^{(01)} \tag{12}$$

For $|\psi\rangle = \frac{1}{\sqrt{3}} |0\rangle + \sqrt{\frac{2}{3}} |1\rangle$:

$$\langle\psi| N_{0+}^\dagger N_{0+} |\psi\rangle = \langle\psi| P_0 P_+ P_0 |\psi\rangle = |\langle 0|\psi\rangle|^2 \cdot |\langle +|0\rangle|^2 = \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{6}$$

Similarly for the other cases, matching the joint probabilities from the previous part.

[2] Use $|0\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle)$ and $|1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$.

ii) Using $P_+^{(\pm)} + P_-^{(\pm)} = \mathbb{1}$:

$$\sum_{i,j} N_{ij}^\dagger N_{ij} = \sum_{i,j} P_i^{(01)} P_j^{(\pm)} P_i^{(01)} = \sum_i P_i^{(01)} \left(\sum_j P_j^{(\pm)} \right) P_i^{(01)}$$
$$= \sum_i P_i^{(01)} \mathbb{1} P_i^{(01)} = \sum_i P_i^{(01)} = \mathbb{1}$$

d) The previous parts showed that a specific cascaded measurement can be described by a single set of measurement operators. Now prove this in general.

Suppose $\{L_\ell\}$ and $\{M_m\}$ are two sets of measurement operators (not necessarily projective). Show that performing measurement $\{L_\ell\}$ followed by measurement $\{M_m\}$ is equivalent to a single measurement with operators $\{N_{\ell m}\}$ where $N_{\ell m} = M_m L_\ell$.

Verify that the $\{N_{\ell m}\}$ satisfy the completeness relation $\sum_{\ell,m} N_{\ell m}^\dagger N_{\ell m} = \mathbb{1}$. Hint[3].

Solution: Probability of joint outcome (ℓ, m):

After measurement $\{L_\ell\}$ on initial state $|\psi\rangle$, outcome ℓ occurs with probability $p(\ell) = \langle\psi| L_\ell^\dagger L_\ell |\psi\rangle$ and the post-measurement state is $|\psi_\ell\rangle = L_\ell |\psi\rangle / \sqrt{p(\ell)}$.

Subsequently measuring with $\{M_m\}$, outcome m occurs with probability:

$$p(m|\ell) = \langle\psi_\ell| M_m^\dagger M_m |\psi_\ell\rangle = \frac{\langle\psi| L_\ell^\dagger M_m^\dagger M_m L_\ell |\psi\rangle}{p(\ell)} \tag{13}$$

The joint probability is:

$$p(\ell, m) = p(\ell) \cdot p(m|\ell) = \langle\psi| L_\ell^\dagger M_m^\dagger M_m L_\ell |\psi\rangle = \langle\psi| N_{\ell m}^\dagger N_{\ell m} |\psi\rangle \tag{14}$$

where $N_{\ell m} = M_m L_\ell$. This is exactly the probability formula for measurement operators $\{N_{\ell m}\}$.

Completeness:

$$\sum_{\ell,m} N_{\ell m}^\dagger N_{\ell m} = \sum_{\ell,m} L_\ell^\dagger M_m^\dagger M_m L_\ell = \sum_\ell L_\ell^\dagger \left(\sum_m M_m^\dagger M_m \right) L_\ell$$
$$= \sum_\ell L_\ell^\dagger \mathbb{1} L_\ell = \sum_\ell L_\ell^\dagger L_\ell = \mathbb{1}$$

where we used the completeness of $\{M_m\}$ and $\{L_\ell\}$.

Problem 3: Observables, Expectation Values, and the Uncertainty Principle (O. Leinonen)

An *observable* is a Hermitian operator M with spectral decomposition:

$$M = \sum_m m\, P_m \tag{15}$$

[3] Use the completeness relations for $\{L_\ell\}$ and $\{M_m\}$.

where m are the eigenvalues (measurement outcomes) and P_m are the projectors onto the corresponding eigenspaces. The expectation value of M in state $|\psi\rangle$ is:

$$\langle M\rangle = \langle\psi|M|\psi\rangle = \sum_m m\, p(m) \tag{16}$$

where $p(m) = \langle\psi|P_m|\psi\rangle$ is the probability of outcome m.

a) Consider the observable σ_z with eigenvalues ± 1.

i) Write the spectral decomposition of σ_z in terms of projectors.

ii) Show that the projectors can be written as $P_\pm = \frac{1}{2}(\mathbb{1} \pm \sigma_z)$.

Solution:

i) The eigenvalues are $+1$ (eigenvector $|0\rangle$) and -1 (eigenvector $|1\rangle$), so:

$$\sigma_z = (+1)\,|0\rangle\langle 0| + (-1)\,|1\rangle\langle 1| = P_+ - P_- \tag{17}$$

where $P_+ = |0\rangle\langle 0|$ and $P_- = |1\rangle\langle 1|$.

ii) From the spectral decomposition and completeness ($P_+ + P_- = \mathbb{1}$):

$$\begin{aligned} P_+ + P_- &= \mathbb{1} \\ P_+ - P_- &= \sigma_z \end{aligned}$$

Adding and subtracting these equations:

$$P_+ = \frac{1}{2}(\mathbb{1} + \sigma_z), \qquad P_- = \frac{1}{2}(\mathbb{1} - \sigma_z) \tag{18}$$

b) For the general qubit state $|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$:

i) Calculate $\langle\sigma_z\rangle$ using the probability formula $\langle\sigma_z\rangle = (+1)p(+) + (-1)p(-)$.

ii) Calculate $\langle\sigma_z\rangle$ directly using $\langle\sigma_z\rangle = \langle\psi|\sigma_z|\psi\rangle$ and verify the results agree.

Solution:

i) Measuring σ_z has outcomes ± 1 with projectors $P_+ = |0\rangle\langle 0|$ and $P_- = |1\rangle\langle 1|$. Thus

$$p(+) = \langle\psi|P_+|\psi\rangle = |\langle 0|\psi\rangle|^2 = \cos^2\frac{\theta}{2}, \qquad p(-) = \langle\psi|P_-|\psi\rangle = |\langle 1|\psi\rangle|^2 = \sin^2\frac{\theta}{2}. \tag{19}$$

Therefore,

$$\langle\sigma_z\rangle = (+1)p(+) + (-1)p(-) = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta. \tag{20}$$

ii) Direct calculation:

$$\begin{aligned} \langle\psi|\sigma_z|\psi\rangle &= \begin{pmatrix}\cos\frac{\theta}{2} & e^{-i\phi}\sin\frac{\theta}{2}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} \\ &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = \cos\theta. \end{aligned}$$

The results agree. Note that $\langle\sigma_z\rangle = r_z$, the z-component of the Bloch vector.

c) The *variance* of an observable M is defined as:

$$(\Delta M)^2 = \langle M^2 \rangle - \langle M \rangle^2 \tag{21}$$

and the *standard deviation* (or uncertainty) is $\Delta M = \sqrt{(\Delta M)^2}$.

Calculate $\Delta\sigma_z$ for $|\psi(\theta,\phi)\rangle$ and interpret the result geometrically on the Bloch sphere. Hint[4].

Solution: Since $\sigma_z^2 = \mathbb{1}$, we have $\langle \sigma_z^2 \rangle = \langle\psi|\mathbb{1}|\psi\rangle = 1$.

Therefore:

$$(\Delta\sigma_z)^2 = 1 - \cos^2\theta = \sin^2\theta \tag{22}$$

$$\Delta\sigma_z = |\sin\theta| \tag{23}$$

Interpretation: The uncertainty is zero ($\Delta\sigma_z = 0$) at the poles ($\theta = 0$ or π), where the state is an eigenstate of σ_z. The uncertainty is maximal ($\Delta\sigma_z = 1$) on the equator ($\theta = \pi/2$), where the probability of each outcome is 1/2. Geometrically, $\sin\theta$ is the distance from the z-axis.

d) (Heisenberg Uncertainty Principle)

For any two observables A and B, the uncertainty principle states:

$$\Delta A \cdot \Delta B \geq \frac{1}{2}\,|\langle [A,B] \rangle| \tag{24}$$

where $[A,B] = AB - BA$ is the commutator.

Consider $A = \sigma_x$ and $B = \sigma_y$ for the state $|0\rangle$.

i) Using $[\sigma_x, \sigma_y] = 2i\sigma_z$, evaluate the right-hand side of the uncertainty relation.

ii) Calculate $\Delta\sigma_x$ and $\Delta\sigma_y$ for $|0\rangle$ and verify the uncertainty relation is satisfied.

Hint[5].

Solution:

i) For $|0\rangle$:

$$\langle [\sigma_x, \sigma_y] \rangle = \langle 0|2i\sigma_z|0\rangle = 2i\,\langle 0|\sigma_z|0\rangle = 2i \cdot (+1) = 2i \tag{25}$$

So the right-hand side is $\frac{1}{2}|2i| = 1$.

ii) Calculate expectation values for $|0\rangle$:

$$\langle \sigma_x \rangle = \langle 0|\sigma_x|0\rangle = 0$$
$$\langle \sigma_y \rangle = \langle 0|\sigma_y|0\rangle = 0$$

Since $\sigma_x^2 = \sigma_y^2 = \mathbb{1}$, we have $\langle \sigma_x^2 \rangle = \langle \sigma_y^2 \rangle = 1$.

Therefore:

$$\Delta\sigma_x = \sqrt{1-0} = 1, \qquad \Delta\sigma_y = \sqrt{1-0} = 1 \tag{26}$$

Check: $\Delta\sigma_x \cdot \Delta\sigma_y = 1 \cdot 1 = 1 \geq 1$ ✓

The bound is saturated! This is a minimum uncertainty state for these observables.

[4] Note that $\sigma_z^2 = \mathbb{1}$.
[5] Use $\sigma_x^2 = \sigma_y^2 = \mathbb{1}$.

e) Find a state for which the uncertainty product $\Delta\sigma_x \cdot \Delta\sigma_y$ is strictly greater than the lower bound. How does the position of this state on the Bloch sphere compare to $|0\rangle$?

Solution: We want $\Delta\sigma_x \cdot \Delta\sigma_y$ to be *strictly* larger than the lower bound

$$\frac{1}{2}\left|\langle[\sigma_x, \sigma_y]\rangle\right| = \frac{1}{2}\left|2i\,\langle\sigma_z\rangle\right| = \left|\langle\sigma_z\rangle\right|.$$

So it is instructive to choose a state with $\langle\sigma_z\rangle \neq 0$ but which is not an eigenstate of σ_x or σ_y.

Consider the Bloch-sphere parametrization

$$|\psi(\theta,\phi)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

and choose, for example, $\theta = \pi/3$ and $\phi = \pi/4$. Then

$$\langle\sigma_z\rangle = \cos\theta = \cos\frac{\pi}{3} = \frac{1}{2},$$

so the lower bound is $|\langle\sigma_z\rangle| = 1/2$.

For a pure qubit state, $\langle\sigma_x\rangle = \sin\theta\cos\phi$ and $\langle\sigma_y\rangle = \sin\theta\sin\phi$. With $\sin(\pi/3) = \sqrt{3}/2$ and $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$, we get

$$\langle\sigma_x\rangle = \langle\sigma_y\rangle = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = \frac{\sqrt{6}}{4}, \qquad \langle\sigma_x\rangle^2 = \langle\sigma_y\rangle^2 = \frac{3}{8}.$$

Using $\sigma_x^2 = \sigma_y^2 = \mathbb{1}$, we have $\langle\sigma_x^2\rangle = \langle\sigma_y^2\rangle = 1$, hence

$$\Delta\sigma_x = \sqrt{1 - \langle\sigma_x\rangle^2} = \sqrt{1 - \frac{3}{8}} = \sqrt{\frac{5}{8}}, \qquad \Delta\sigma_y = \sqrt{\frac{5}{8}}.$$

Therefore,

$$\Delta\sigma_x \cdot \Delta\sigma_y = \frac{5}{8} > \frac{1}{2},$$

which is *strictly* larger than the lower bound.

Geometric interpretation: Compared to $|0\rangle$ (north pole), this state is tilted away from the z-axis (so $\langle\sigma_z\rangle$ decreases from 1 to 1/2), and it also has nonzero x- and y-components, meaning it is not sharply defined in either the x- or y-measurement.

Problem 4: Bell Measurement and Superdense Coding (J. Gulla)

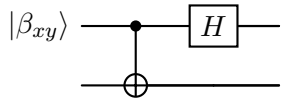
A *Bell measurement* is a projective measurement in the Bell basis $\{|\beta_{00}\rangle, |\beta_{01}\rangle, |\beta_{10}\rangle, |\beta_{11}\rangle\}$, where

$$|\beta_{00}\rangle = \left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\beta_{10}\rangle = \left|\Phi^-\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$

$$|\beta_{01}\rangle = \left|\Psi^+\right\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\beta_{11}\rangle = \left|\Psi^-\right\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}.$$

In the previous exercise you showed that the circuit $U_{\text{prep}} = \text{CNOT}(H \otimes \mathbb{1})$ maps $|xy\rangle \mapsto |\beta_{xy}\rangle$.

a) (**Bell decoding**) Show that the inverse circuit maps Bell states back to computational basis states:



i.e., $\quad |\beta_{xy}\rangle \mapsto |xy\rangle.$

Solution: Let $U_{\text{prep}} = \text{CNOT}(H \otimes \mathbb{1})$. Since $H^\dagger = H$ and $\text{CNOT}^\dagger = \text{CNOT}$,

$$U_{\text{prep}}^\dagger = (H \otimes \mathbb{1})\,\text{CNOT}.$$

The displayed circuit is exactly $U_{\text{prep}}^\dagger$, so it must undo the preparation:

$$U_{\text{prep}}^\dagger |\beta_{xy}\rangle = U_{\text{prep}}^\dagger U_{\text{prep}} |xy\rangle = |xy\rangle.$$

(If desired, this can be verified explicitly by applying CNOT and then H to each Bell state.)

b) (**Bell measurement**) Explain why applying the decoding circuit above and then measuring both qubits in the computational basis implements a Bell measurement, i.e. measures in the Bell basis.

Solution: A measurement in the computational basis after a unitary U is equivalent to a measurement in the orthonormal basis $\{U^\dagger |00\rangle, U^\dagger |01\rangle, U^\dagger |10\rangle, U^\dagger |11\rangle\}$ before U.

Here $U = U_{\text{prep}}^\dagger$, so $U^\dagger = U_{\text{prep}}$ and

$$U^\dagger |xy\rangle = U_{\text{prep}} |xy\rangle = |\beta_{xy}\rangle.$$

Thus, decoding followed by computational-basis measurement projects onto the Bell states and reports the label (x, y). Equivalently, if the input is exactly $|\beta_{xy}\rangle$, then after decoding it becomes $|xy\rangle$ deterministically.

c) (**Superdense coding**) Alice and Bob share the entangled state $|\Phi^+\rangle = |\beta_{00}\rangle$. Alice wants to send two classical bits $(x, y) \in \{0, 1\}^2$ to Bob by sending *one* qubit.

- **i)** Show that if Alice applies the single-qubit operation $\sigma_z^x \sigma_x^y$ to *her* qubit, the joint state becomes $|\beta_{xy}\rangle$ (up to a global phase).
- **ii)** Explain how Bob can recover (x, y) with certainty using a Bell measurement, and draw the decoding circuit.

Solution:

i) Start from $|\beta_{00}\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Applying Pauli operators to the first qubit gives:

$$(\mathbb{1} \otimes \mathbb{1}) \left|\Phi^+\right\rangle = |\beta_{00}\rangle, \qquad (\sigma_x \otimes \mathbb{1}) \left|\Phi^+\right\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}} = \left|\Psi^+\right\rangle = |\beta_{01}\rangle,$$

$$(\sigma_z \otimes \mathbb{1}) \left|\Phi^+\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} = \left|\Phi^-\right\rangle = |\beta_{10}\rangle.$$

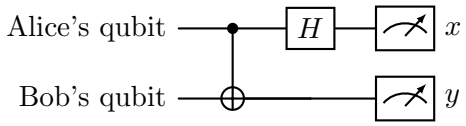
For the remaining case,

$$(\sigma_z \sigma_x \otimes \mathbb{1}) \left|\Phi^+\right\rangle = (\sigma_z \otimes \mathbb{1}) \left|\Psi^+\right\rangle = \frac{\sigma_z |0\rangle |1\rangle + \sigma_z |1\rangle |0\rangle}{\sqrt{2}} = \frac{|01\rangle - |10\rangle}{\sqrt{2}} = \left|\Psi^-\right\rangle = |\beta_{11}\rangle,$$

(up to an irrelevant global phase if one instead uses σ_y). Thus, for bits x, y, the operation $\sigma_z^x \sigma_x^y$ maps $|\beta_{00}\rangle$ to $|\beta_{xy}\rangle$ (up to global phase).

ii) After Alice sends her qubit to Bob, Bob holds both qubits and performs a Bell measurement. Using the decoding circuit from the first part, he maps $|\beta_{xy}\rangle \mapsto |xy\rangle$ and then measures in the computational basis, thereby learning (x, y) with certainty.

A standard decoding circuit is:



(up to the convention of which output wire is labeled x or y).